Post-Quantum Signatures from MPC in the Head

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PQ-TLS Summer School

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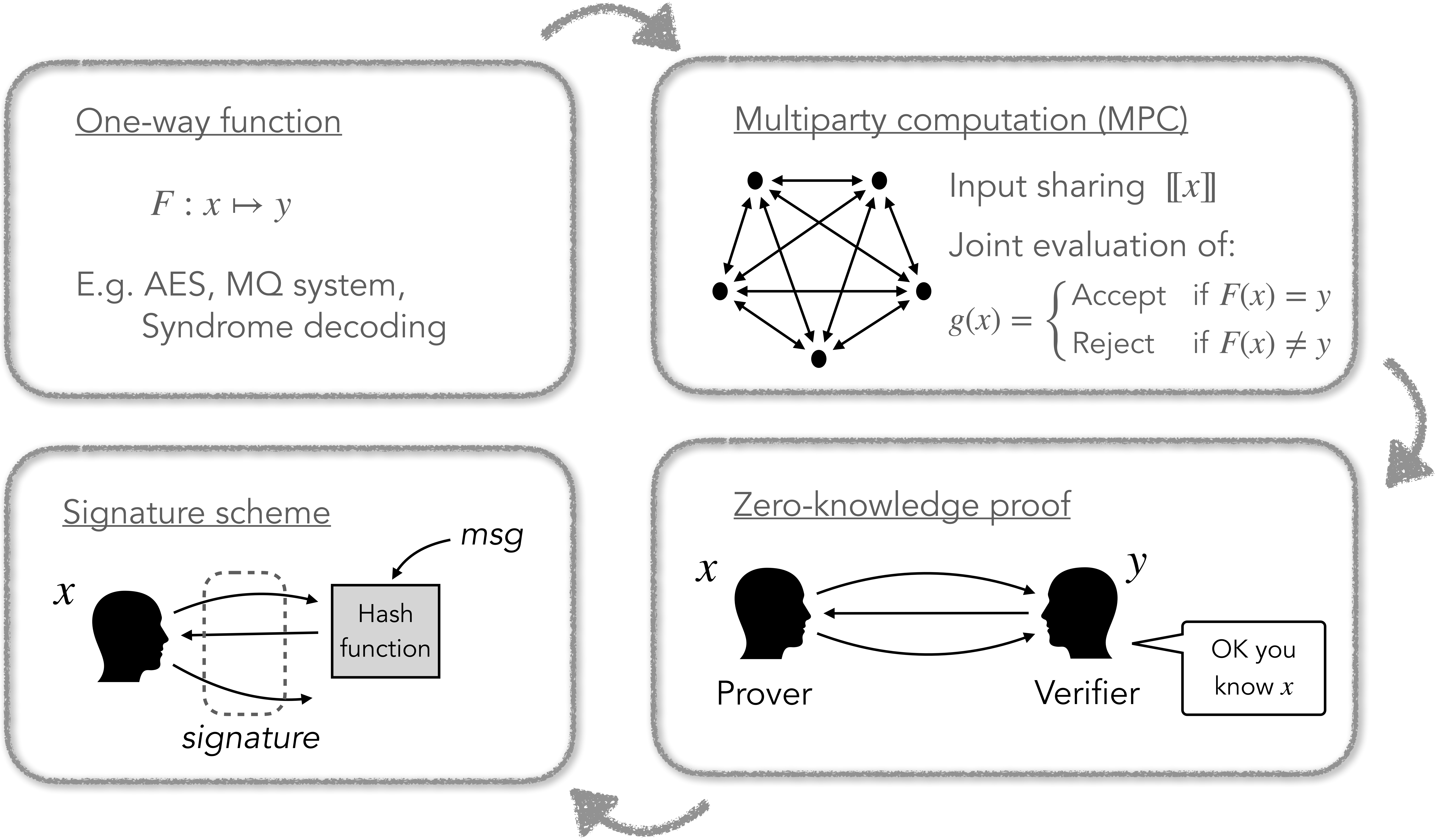
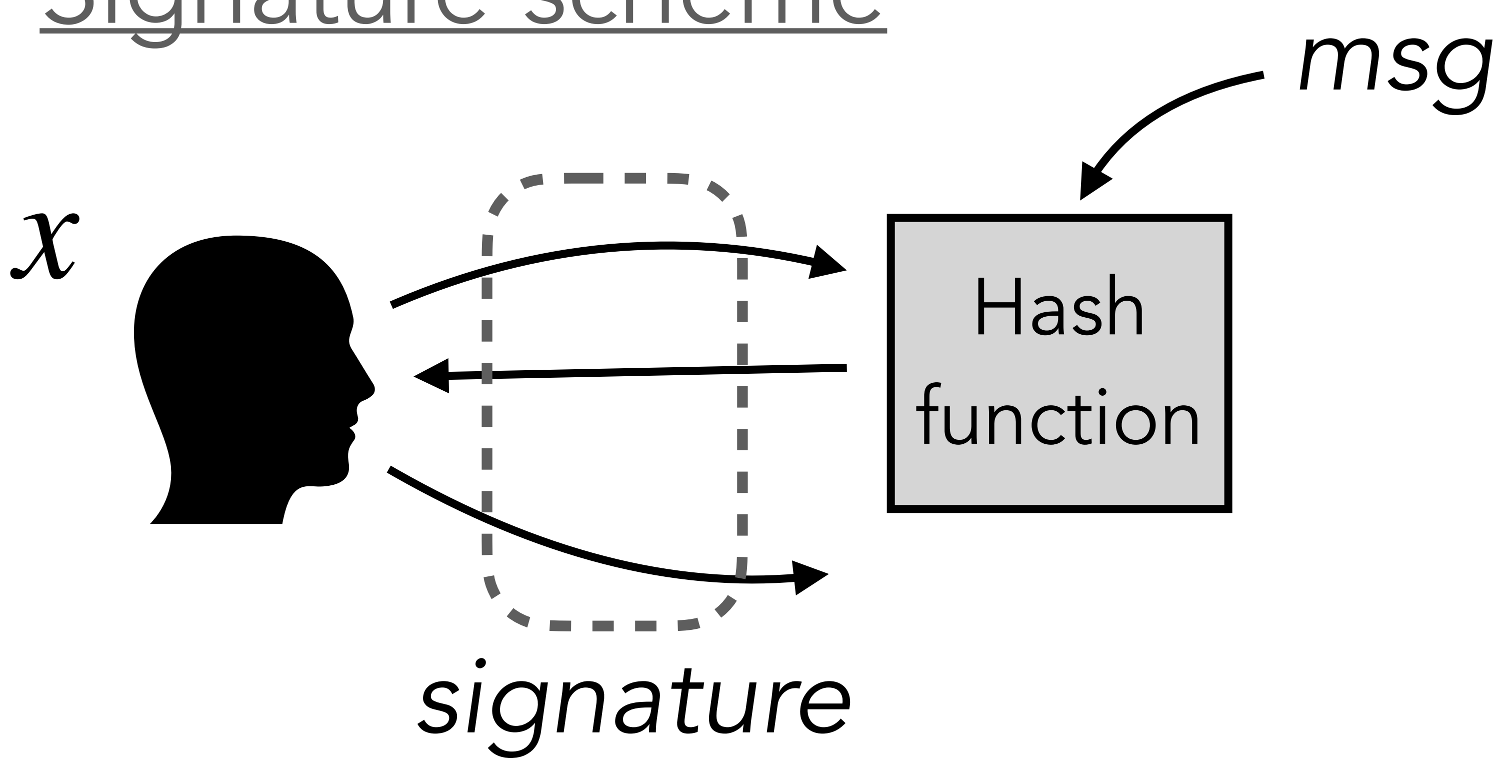

One-way function
$F : x \mapsto y$
E.g. AES, MQ system,
Syndrome decoding
Multiparty computation (MPC)
Input sharing $[\![x]\!]$
Joint evaluation of:
$g(x) = \begin{cases} \text{Accept} & \text{if } F(x) = y \\ \text{Reject} & \text{if } F(x) \neq y \end{cases}$
Zero-knowledge proof
x
y
Prover
Verifier
OK you know x
Signature scheme
x
msg
Hash function
signature

One-way function

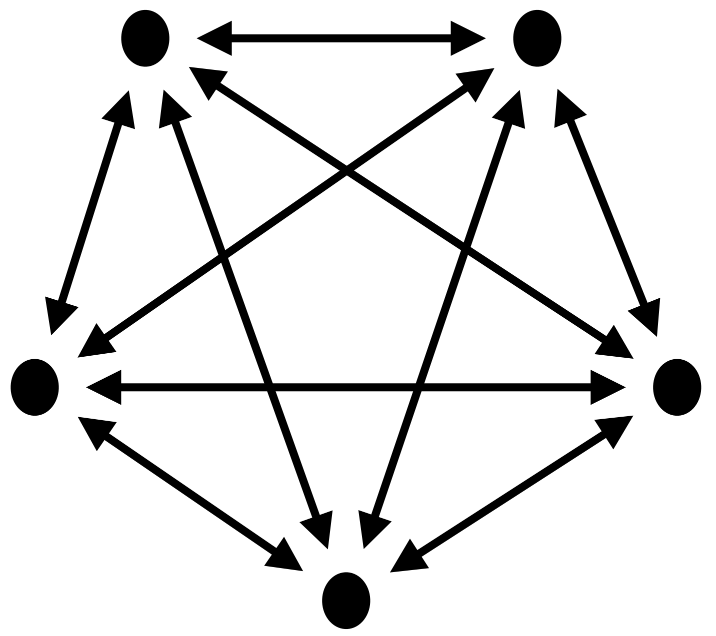
$$F : x \mapsto y$$

E.g. AES, MQ system,
Syndrome decoding

Signature scheme



Multiparty computation (MPC)



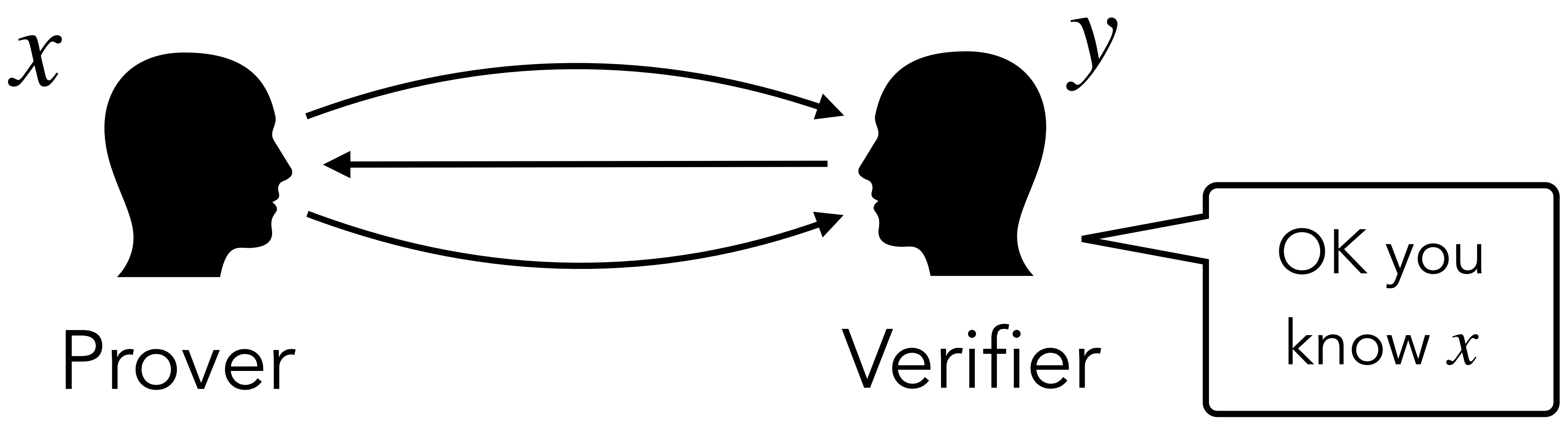
Input sharing $[\![x]\!]$

Joint evaluation of:

$$g(x) = \begin{cases} \text{Accept} & \text{if } F(x) = y \\ \text{Reject} & \text{if } F(x) \neq y \end{cases}$$

MPC in the Head

Zero-knowledge proof

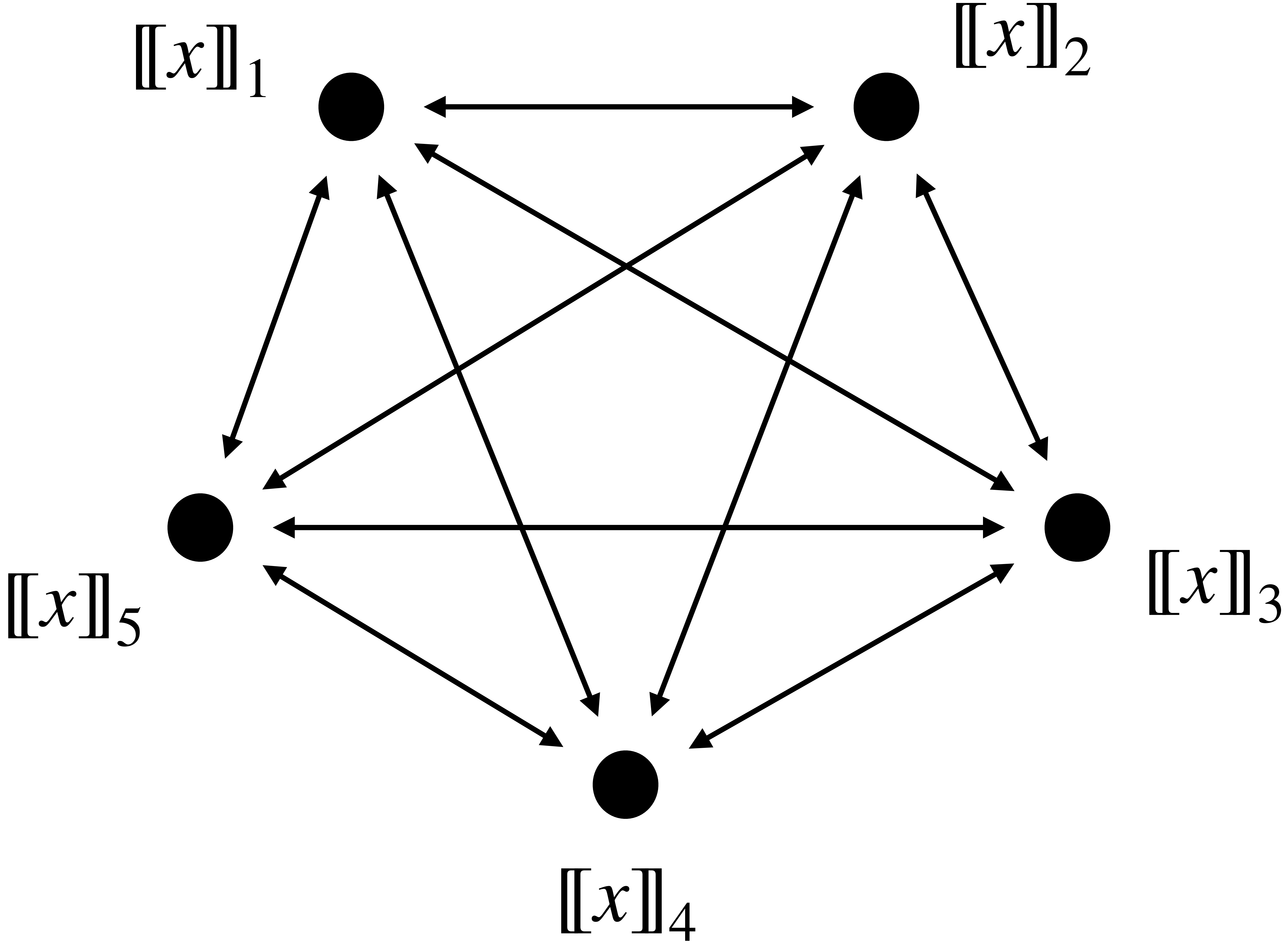


Roadmap

- MPC-in-the-Head with Additive Secret Sharing
- Optimisations
- SDitH Signature Scheme: MPCitH with Syndrome Decoding
- MPC-in-the-Head with Threshold Secret Sharing

MPC-in-the-Head with Additive Secret Sharing

MPC model

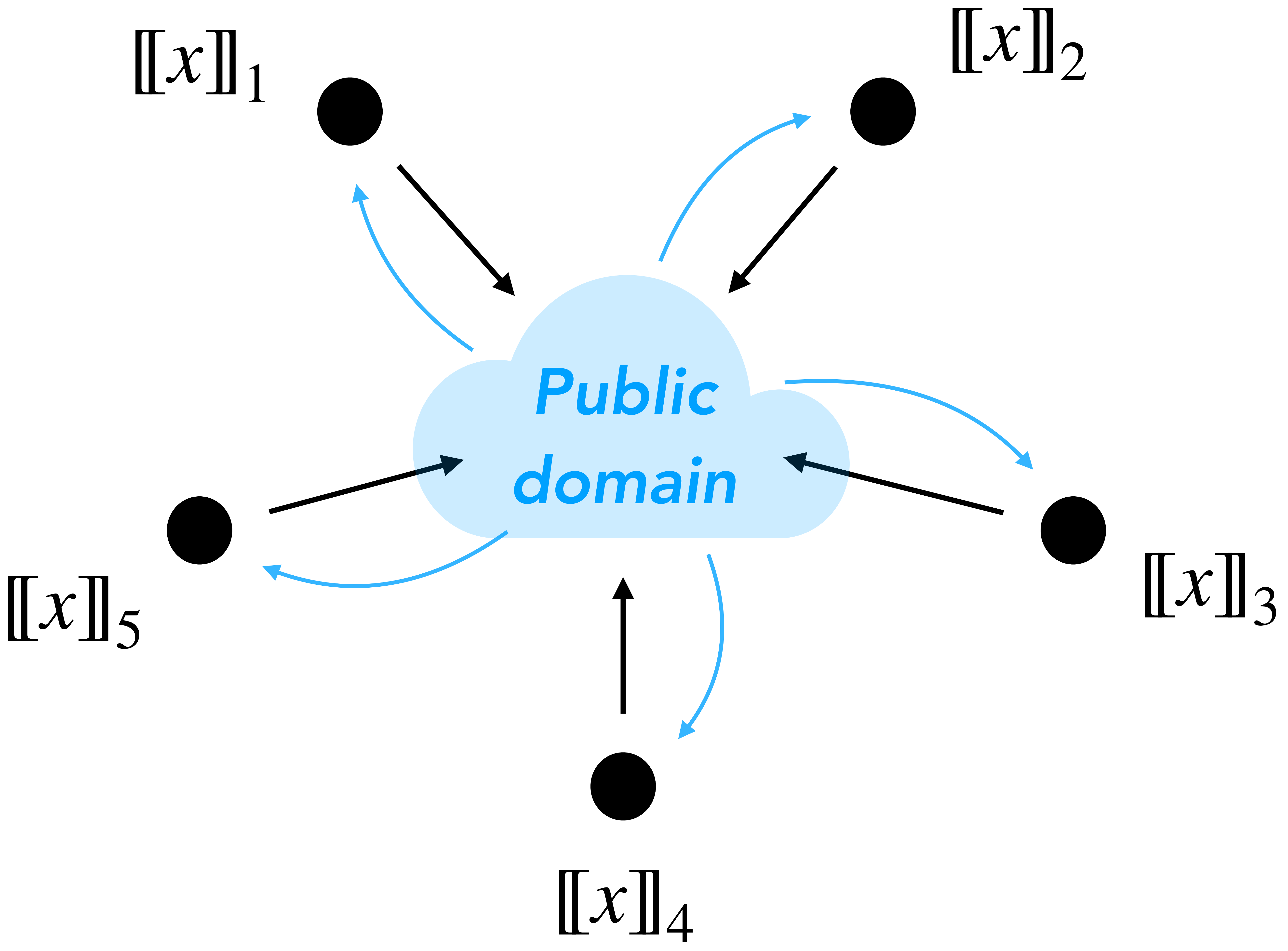


- **Jointly compute**

$$g(x) = \begin{cases} \text{Accept} & \text{if } F(x) = y \\ \text{Reject} & \text{if } F(x) \neq y \end{cases}$$

- $(N-1)$ **private:** the views of any $N-1$ parties provide no information on x
- **Semi-honest model:** assuming that the parties follow the steps of the protocol

MPC model



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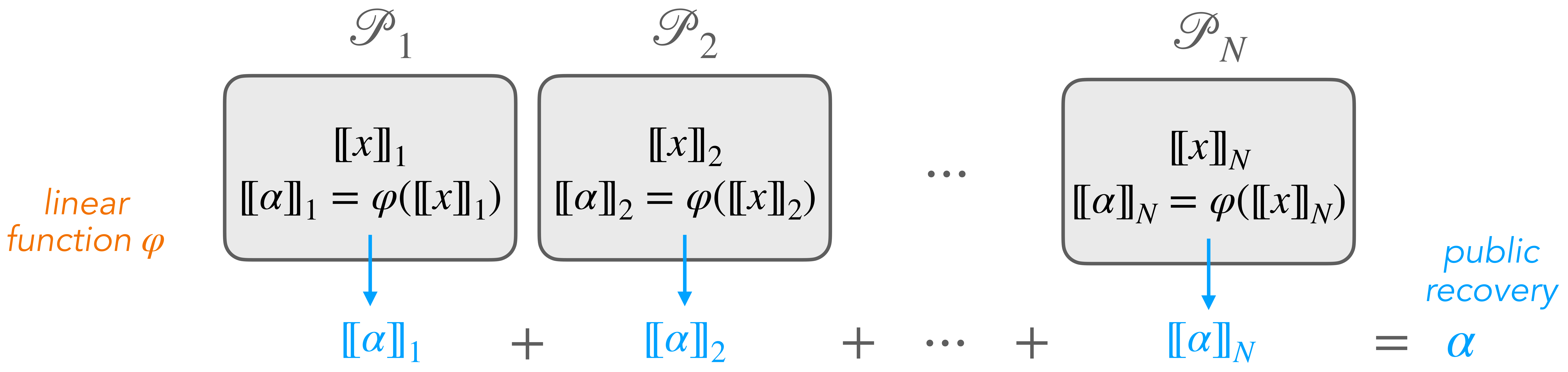
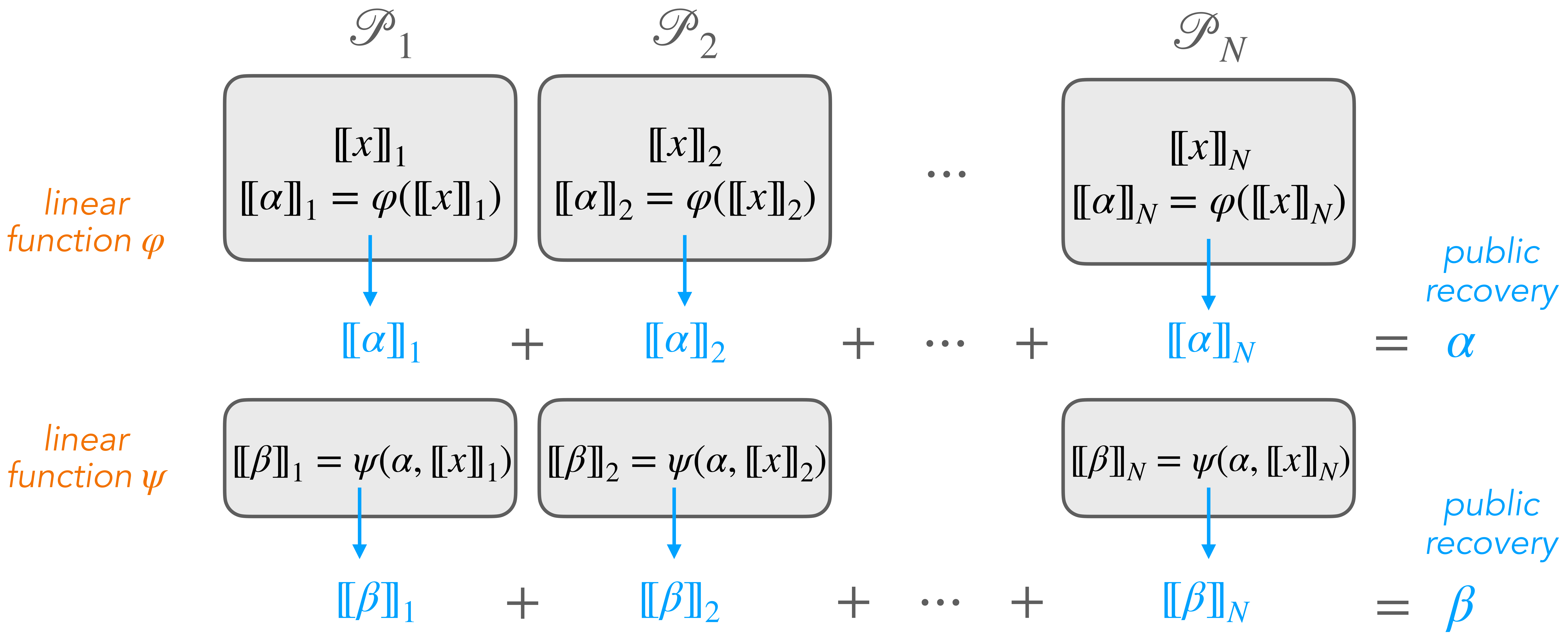
$$g(x) = \begin{cases} \text{Accept} & \text{if } F(x) = y \\ \text{Reject} & \text{if } F(x) \neq y \end{cases}$$

- $(N-1)$ **private:** the views of any $N-1$ parties provide no information on x
- **Semi-honest model:** assuming that the parties follow the steps of the protocol
- ***Broadcast model***
  - *Parties locally compute on their shares* $[\![x]\!] \mapsto [\![\alpha]\!]$
  - *Parties broadcast* $[\![\alpha]\!]$ *and recompute* α
  - *Parties start again (now knowing* α*)*


$\mathscr{P}_1$
$\mathscr{P}_2$
$\mathscr{P}_N$
$[\![x]\!]_1$
$[\![\alpha]\!]_1 = \varphi([\![x]\!]_1)$
$[\![x]\!]_2$
$[\![\alpha]\!]_2 = \varphi([\![x]\!]_2)$
$\cdots$
$[\![x]\!]_N$
$[\![\alpha]\!]_N = \varphi([\![x]\!]_N)$
linear function φ
public recovery
$[\![\alpha]\!]_1 + [\![\alpha]\!]_2 + \cdots + [\![\alpha]\!]_N = \alpha$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public recovery
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linear function ψ
$[\![\beta]\!]_1 = \psi(\alpha, [\![x]\!]_1)$
$[\![\beta]\!]_2 = \psi(\alpha, [\![x]\!]_2)$
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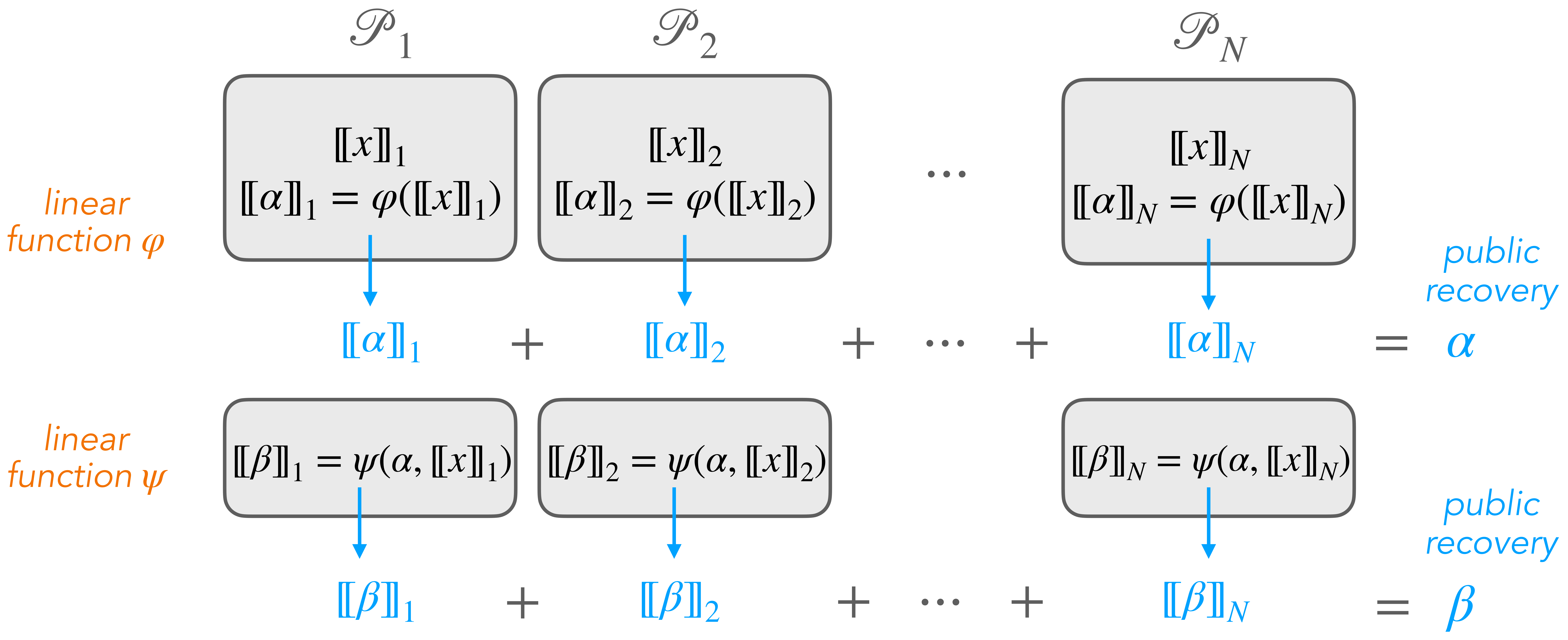
$[\![\beta]\!]_1 + [\![\beta]\!]_2 + \cdots + [\![\beta]\!]_N = \beta$



and so on…

$$g : (y, \alpha, \beta, \ldots) \mapsto \begin{cases} \text{Accept} \\ \text{Reject} \end{cases}$$

Example: matrix multiplication $y = Hx$

$\mathscr{P}_1$ | $\mathscr{P}_2$ | $\cdots$ | $\mathscr{P}_N$

$[\![x]\!]_1$, $[\![\alpha]\!]_1 = H \cdot [\![x]\!]_1$

$[\![x]\!]_2$, $[\![\alpha]\!]_2 = H \cdot [\![x]\!]_2$

$[\![x]\!]_N$, $[\![\alpha]\!]_N = H \cdot [\![x]\!]_N$

mult. by H is linear

public recovery

$$[\![\alpha]\!]_1 + [\![\alpha]\!]_2 + \cdots + [\![\alpha]\!]_N = \alpha$$

$$g(y, \alpha) = \begin{cases} \text{Accept} & \text{if } y = \alpha \\ \text{Reject} & \text{if } y \neq \alpha \end{cases}$$

$$g(y, \alpha) = \text{Accept} \iff Hx = y$$

MPCitH transform

Prover

Verifier

MPCitH transform

① Generate and commit shares

$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$

$\mathrm{Com}^{\rho_1}([\![x]\!]_1)$

$\cdots$

$\mathrm{Com}^{\rho_N}([\![x]\!]_N)$

$\longrightarrow$

Prover

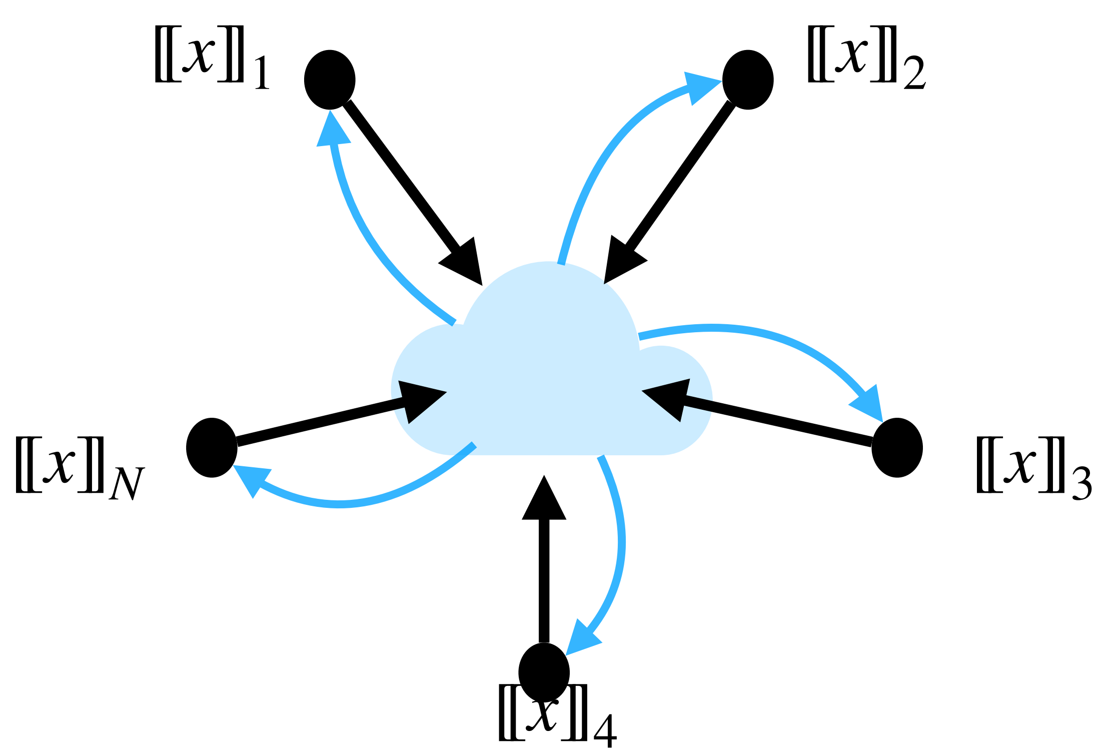
Verifier

MPCitH transform

① Generate and commit shares

$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$

② Run MPC in their head



$\mathrm{Com}^{\rho_1}([\![x]\!]_1)$

$\cdots$

$\mathrm{Com}^{\rho_N}([\![x]\!]_N)$

send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$

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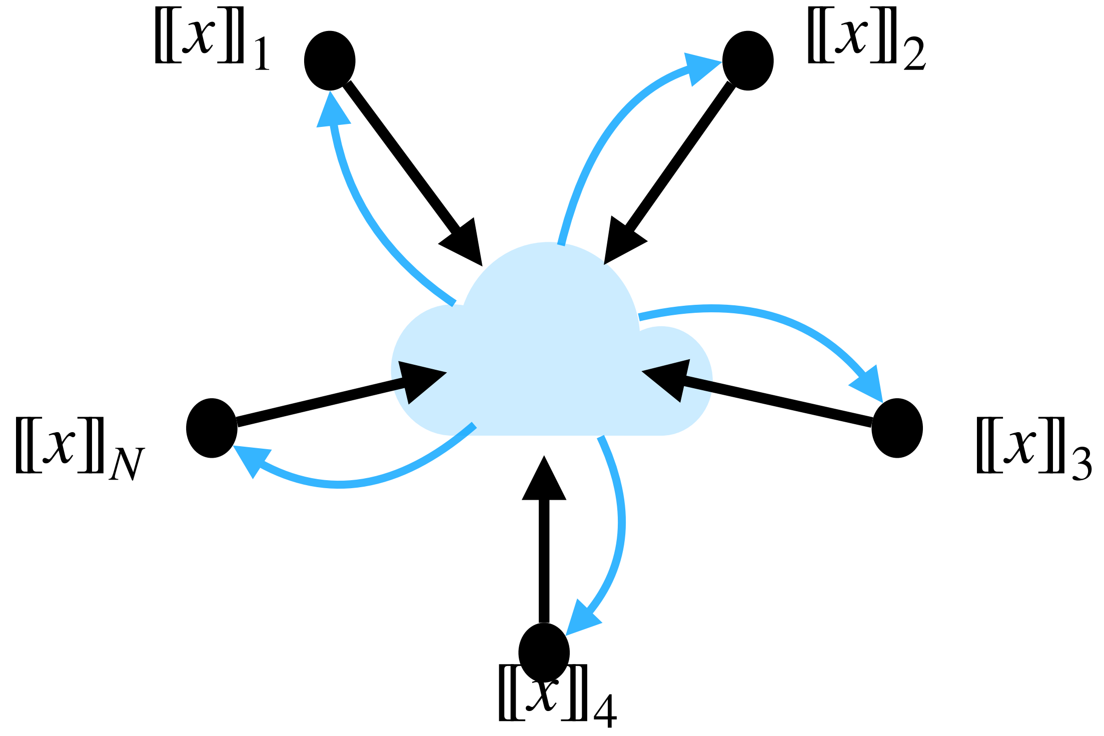

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$\cdots$

$\mathrm{Com}^{\rho_N}([\![x]\!]_N)$

$\longrightarrow$

send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$

$\longrightarrow$

③ Chose a random party

$i^* \xleftarrow{\$} \{1,\ldots,N\}$

i^*

$\longleftarrow$

Prover

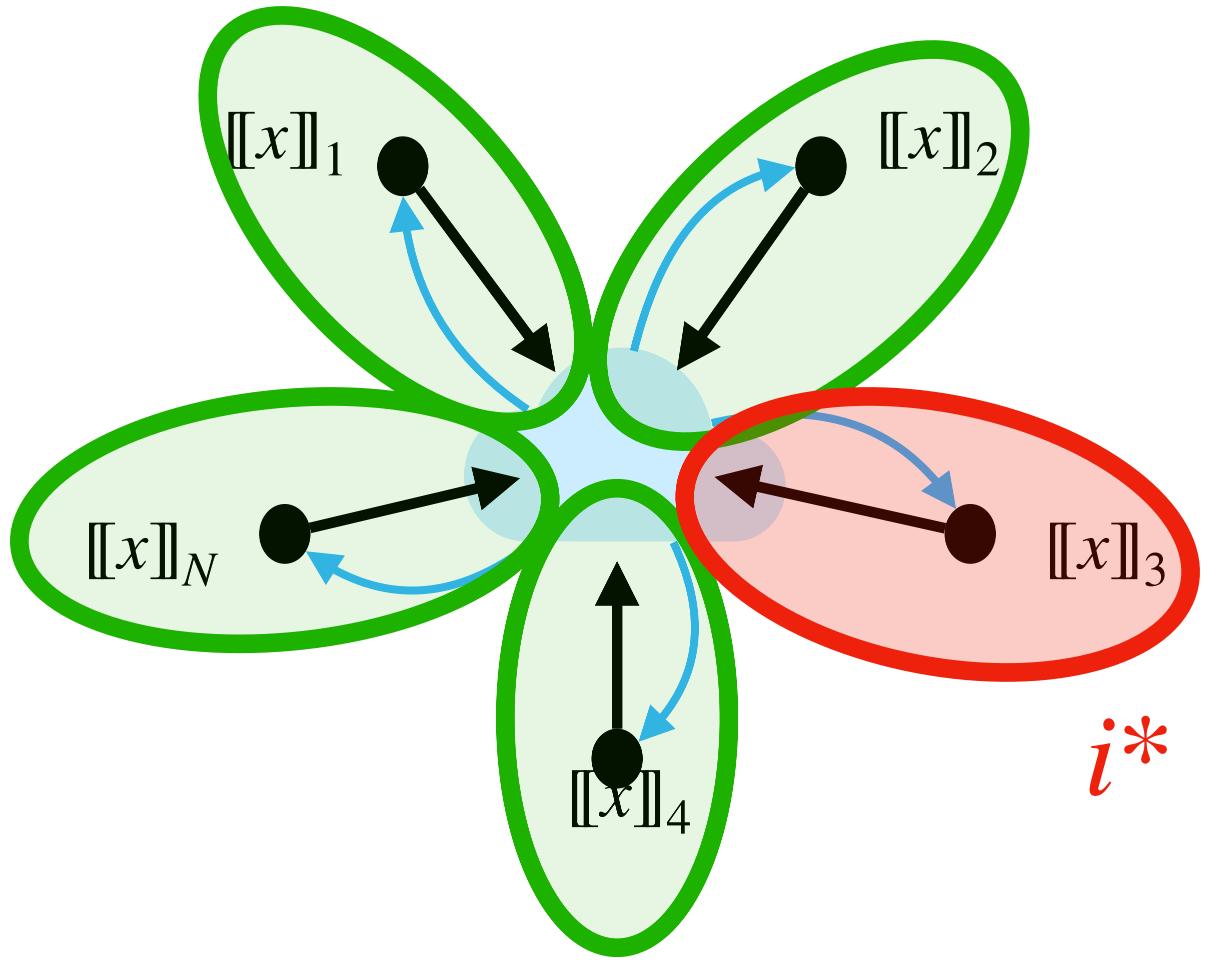
Verifier

MPCitH transform

① Generate and commit shares

$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$

② Run MPC in their head



④ Open parties $\{1,\ldots,N\}\backslash\{i^*\}$

$\mathrm{Com}^{\rho_1}([\![x]\!]_1)$

$\cdots$

$\mathrm{Com}^{\rho_N}([\![x]\!]_N)$ →

send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$ →

← i^*

$([\![x]\!]_i, \rho_i)_{i \neq i^*}$ →

③ Chose a random party

$i^* \xleftarrow{\$} \{1,\ldots,N\}$

Prover

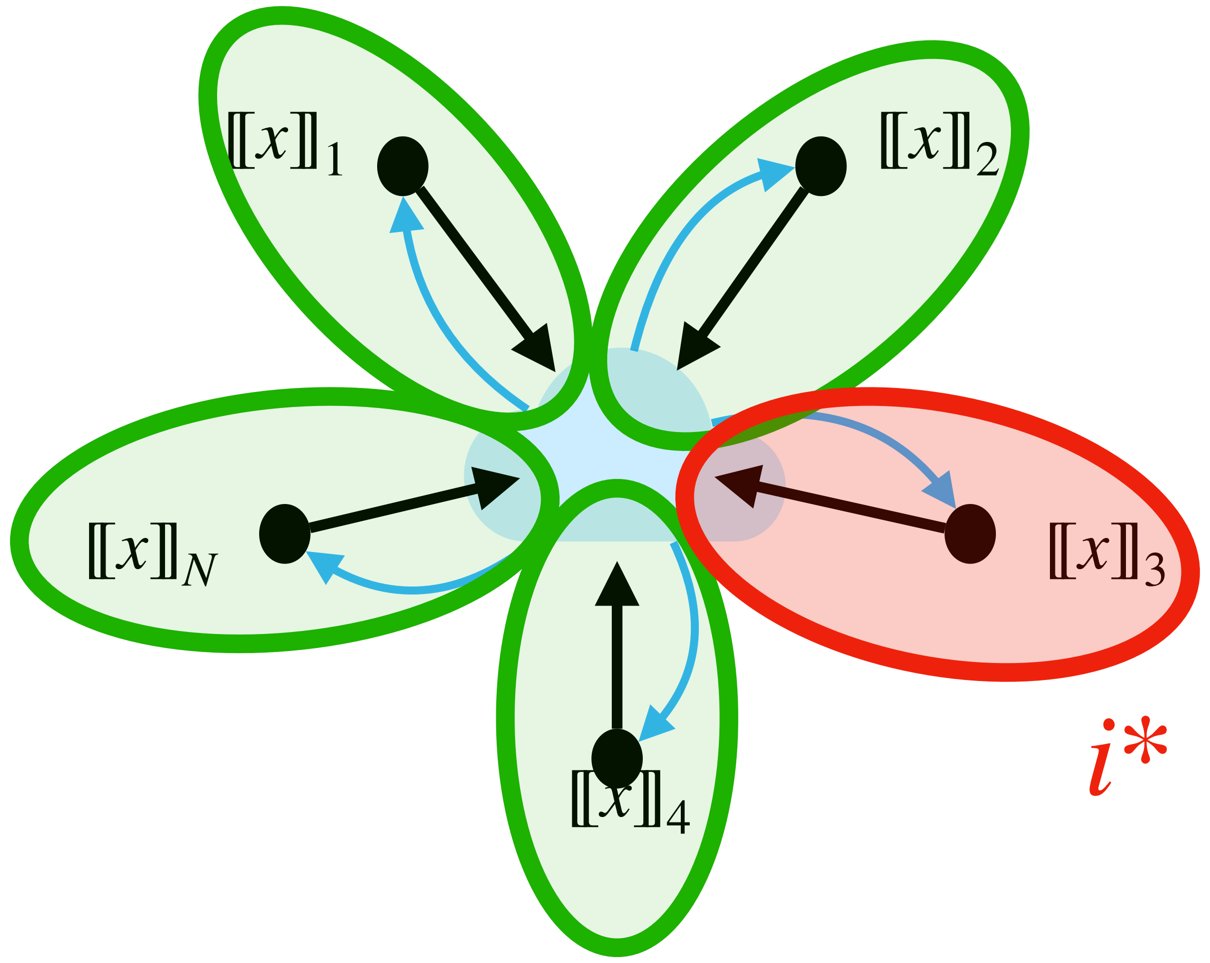
Verifier

MPCitH transform

① Generate and commit shares

$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$

② Run MPC in their head



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$\cdots$

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send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$

i^*

$([\![x]\!]_i, \rho_i)_{i \neq i^*}$

③ Chose a random party

$i^* \xleftarrow{\$} \{1,\ldots,N\}$

⑤ Check $\forall i \neq i^*$

- Commitments $\mathrm{Com}^{\rho_i}([\![x]\!]_i)$
- MPC computation $[\![\alpha]\!]_i = \varphi([\![x]\!]_i)$

Check $g(y, \alpha) =$ Accept

Prover

Verifier

MPCitH transform

- **Zero-knowledge** $\iff$ MPC protocol is $(N-1)$-private

MPCitH transform

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- **Soundness**
  - if $g(y, \alpha) \neq \text{Accept}$ → Verifier rejects

MPCitH transform

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  - if $g(y, \alpha) =$ Accept, then
    - either $[\![x]\!]$ = sharing of correct witness $F(x) = y$

      → Prover honest

MPCitH transform

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      → Prover honest
    - or Prover has cheated for at least one party

      → Cheat undetected with proba $\dfrac{1}{N}$

MPCitH transform

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$\boxed{\frac{1}{N}}$

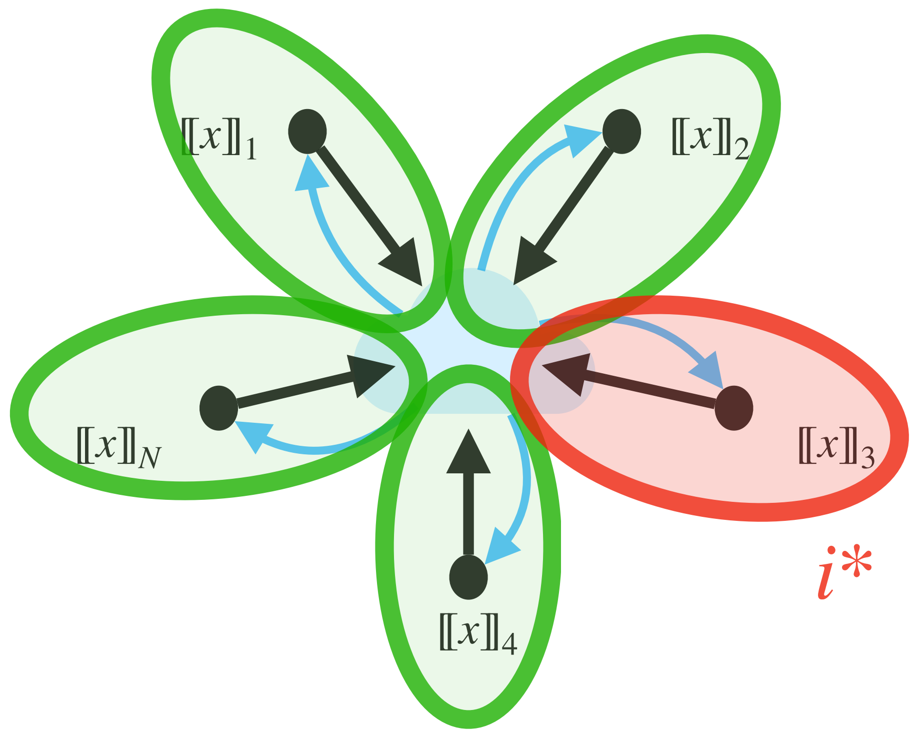
Soundness error

Example: matrix multiplication $y = Hx$

$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_N$
$[\![x]\!]_1$ $[\![\alpha]\!]_1 = H \cdot [\![x]\!]_1$	$[\![x]\!]_2$ $[\![\alpha]\!]_2 = H \cdot [\![x]\!]_2$	$\cdots$	$[\![x]\!]_N$ $[\![\alpha]\!]_N = H \cdot [\![x]\!]_N$

$$[\![\alpha]\!]_1 \; + \; [\![\alpha]\!]_2 \; + \cdots + \; [\![\alpha]\!]_N \; = \; \alpha$$

Prover



$$\xrightarrow{\{\mathrm{Com}^{\rho_i}([\![x]\!]_i)\}}$$

$$\xrightarrow{\{[\![\alpha]\!]_i\}}$$

$$\xleftarrow{i^*}$$

$$\xrightarrow{\{[\![x]\!]_i, \rho_i\}_{i \neq i^*}}$$

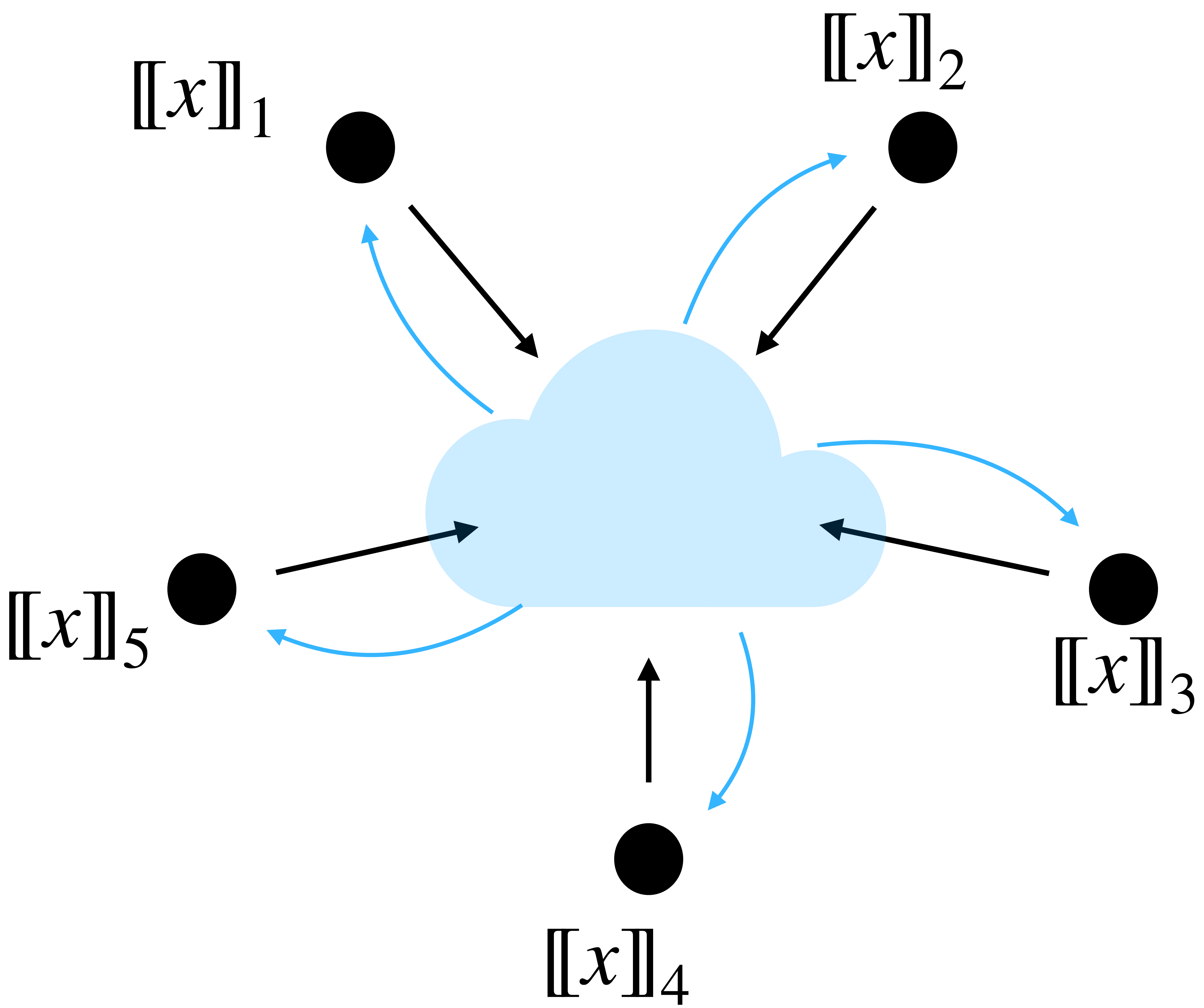
Verifier

Check $\forall i \neq i^*$

- Commitments $\mathrm{Com}^{\rho_i}([\![x]\!]_i)$
- MPC computation $[\![\alpha]\!]_i = H \cdot [\![x]\!]_i$

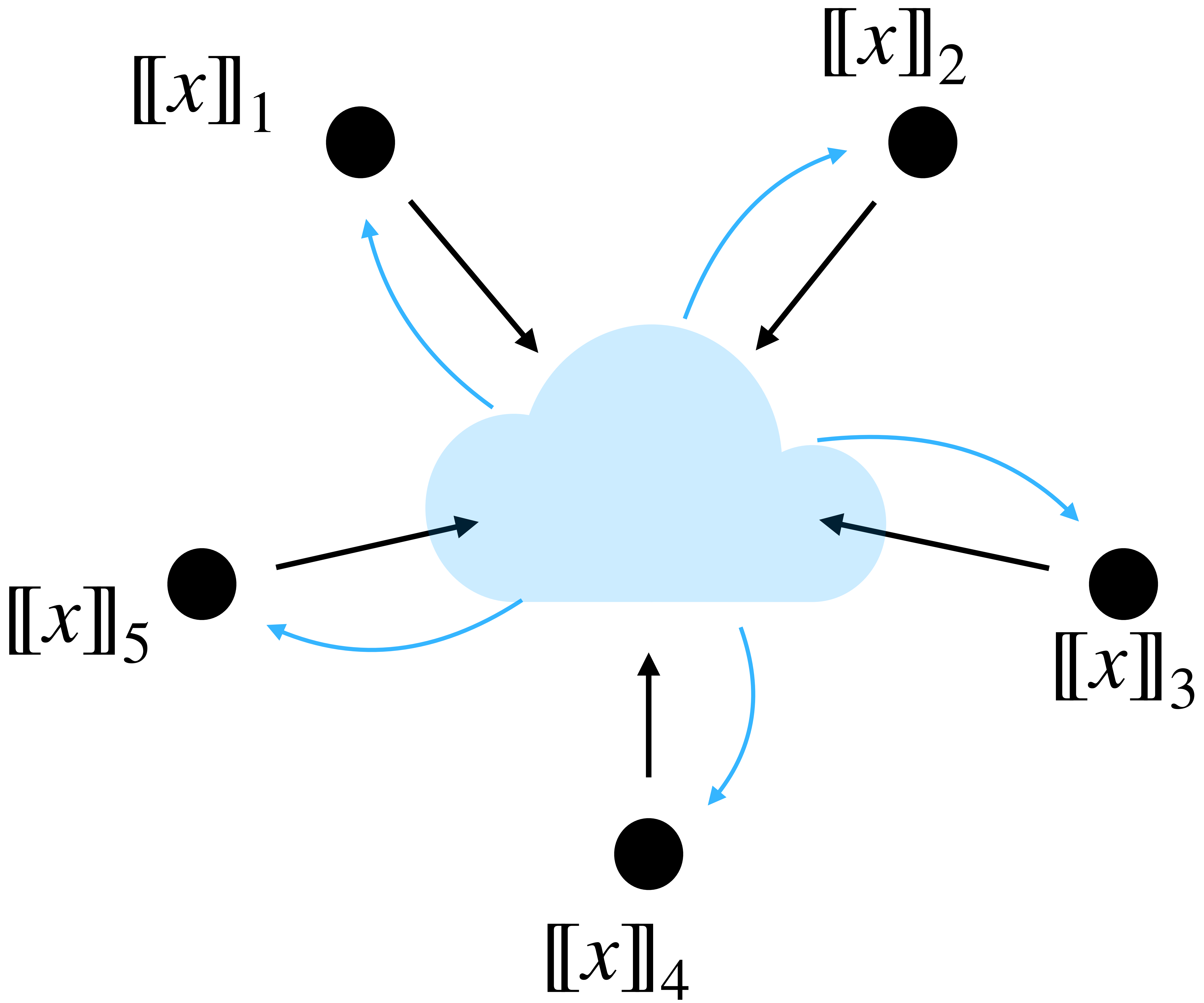
Check $\alpha := \Sigma_i [\![\alpha]\!]_i \; = \; y$

Complete MPC model

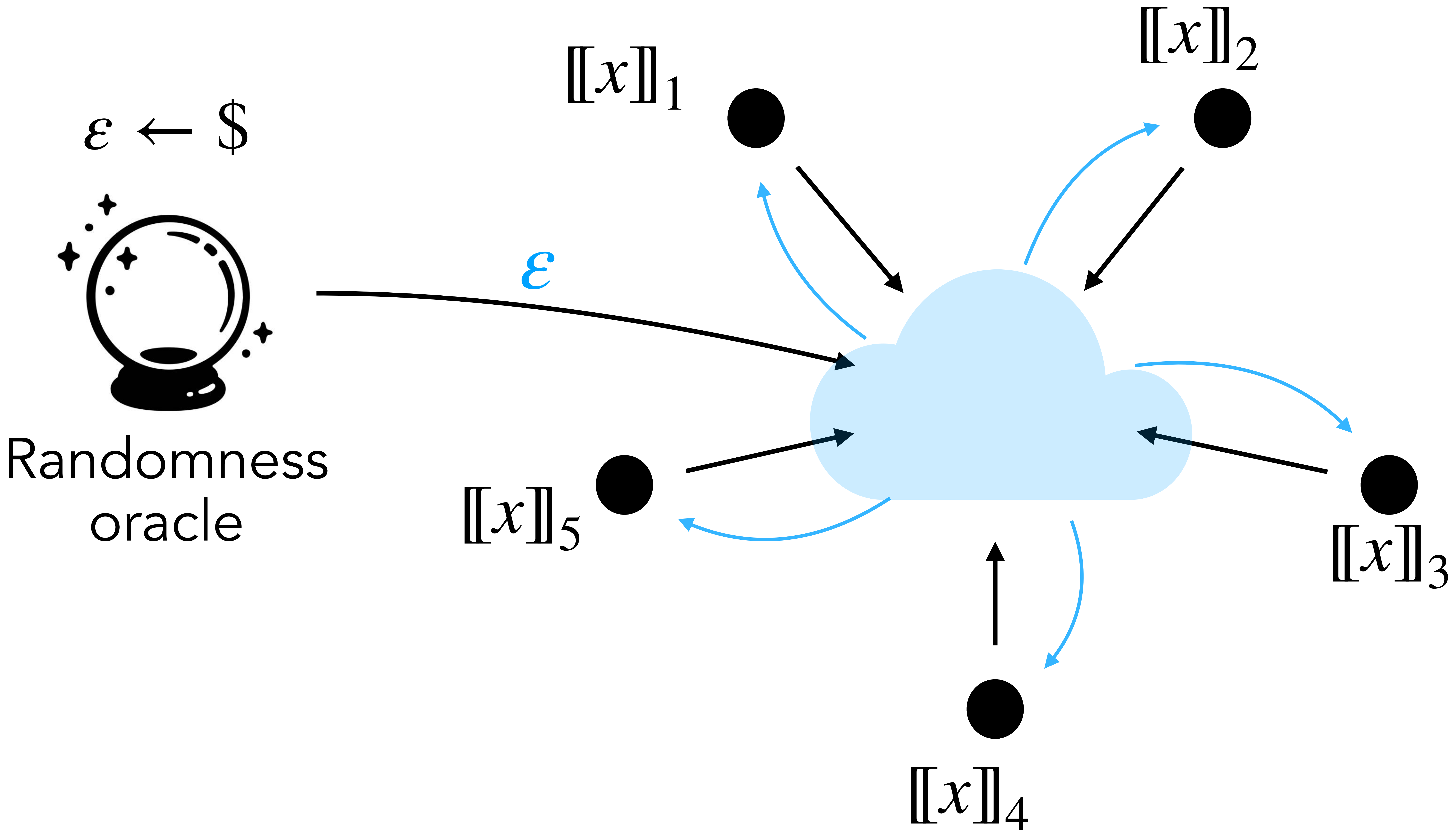


Complete MPC model

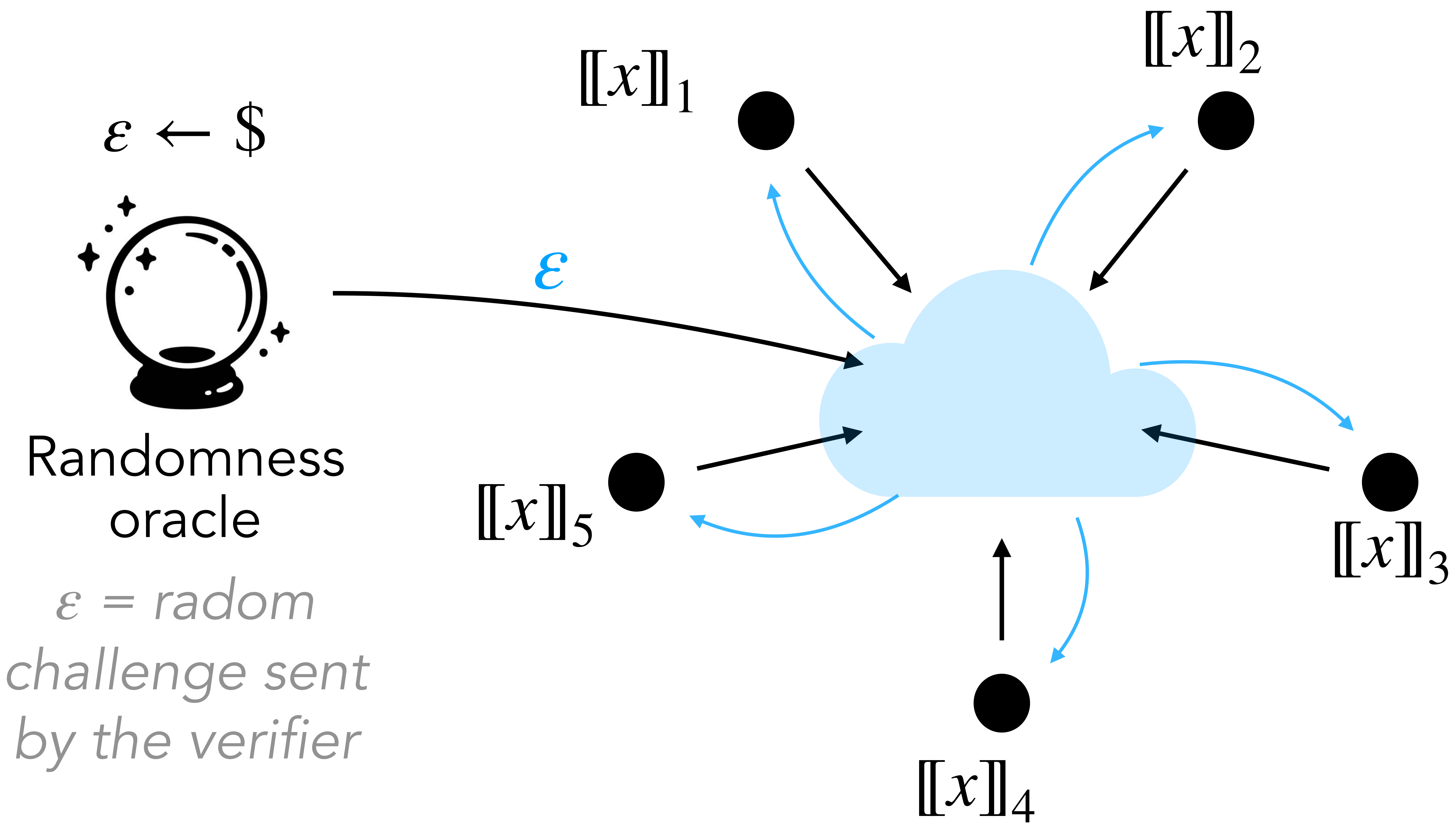
Randomness
oracle



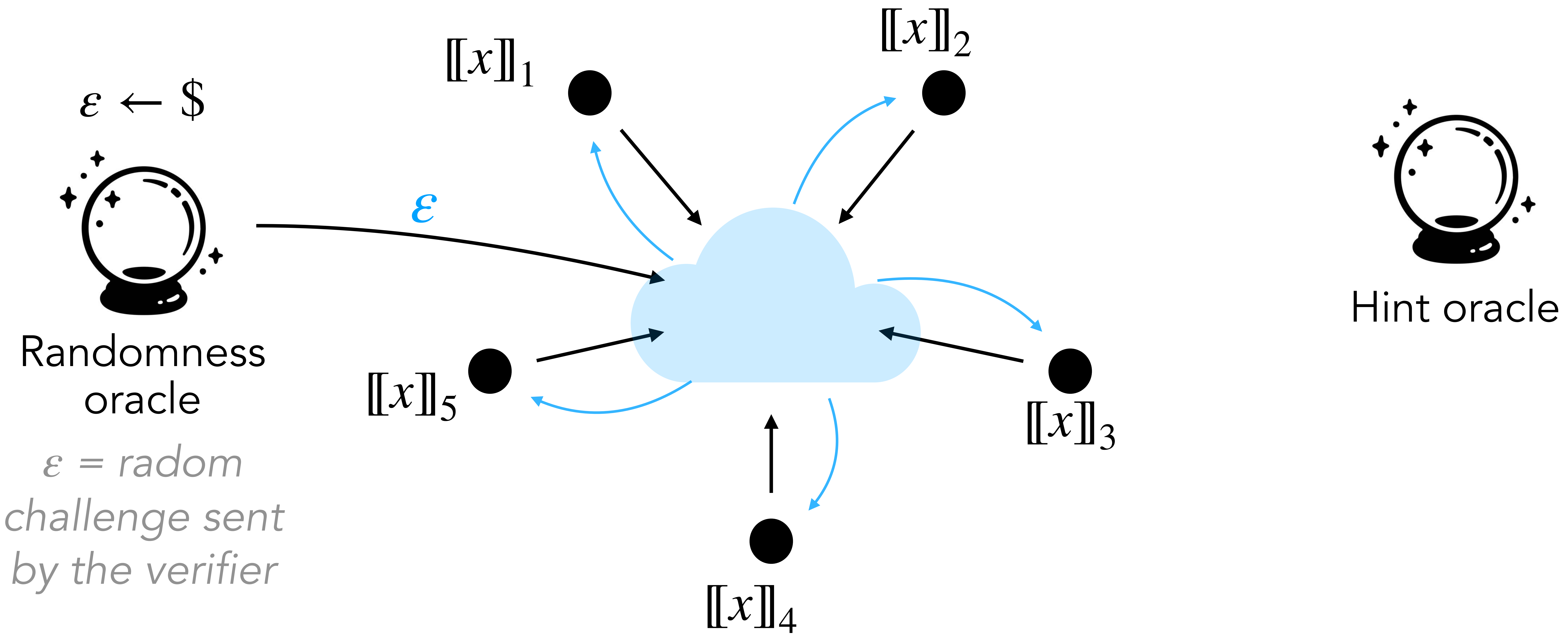
Complete MPC model



Complete MPC model



Complete MPC model



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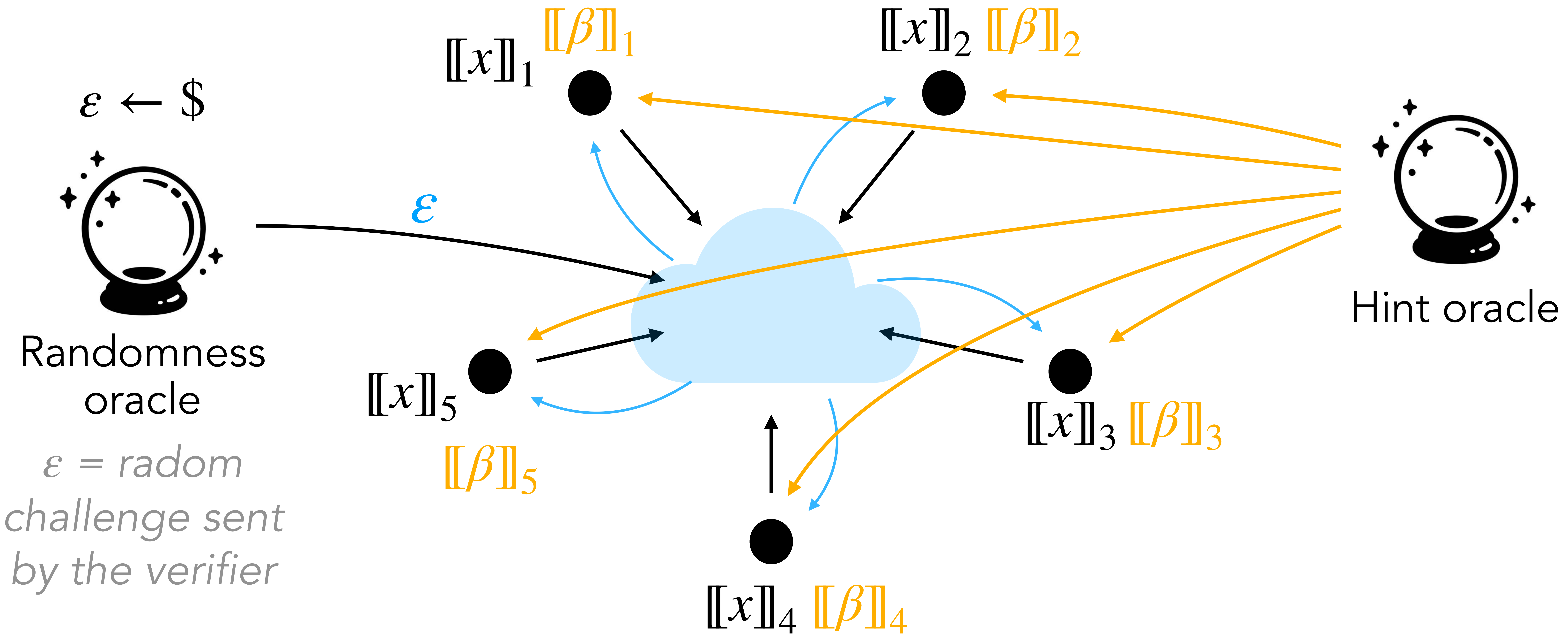

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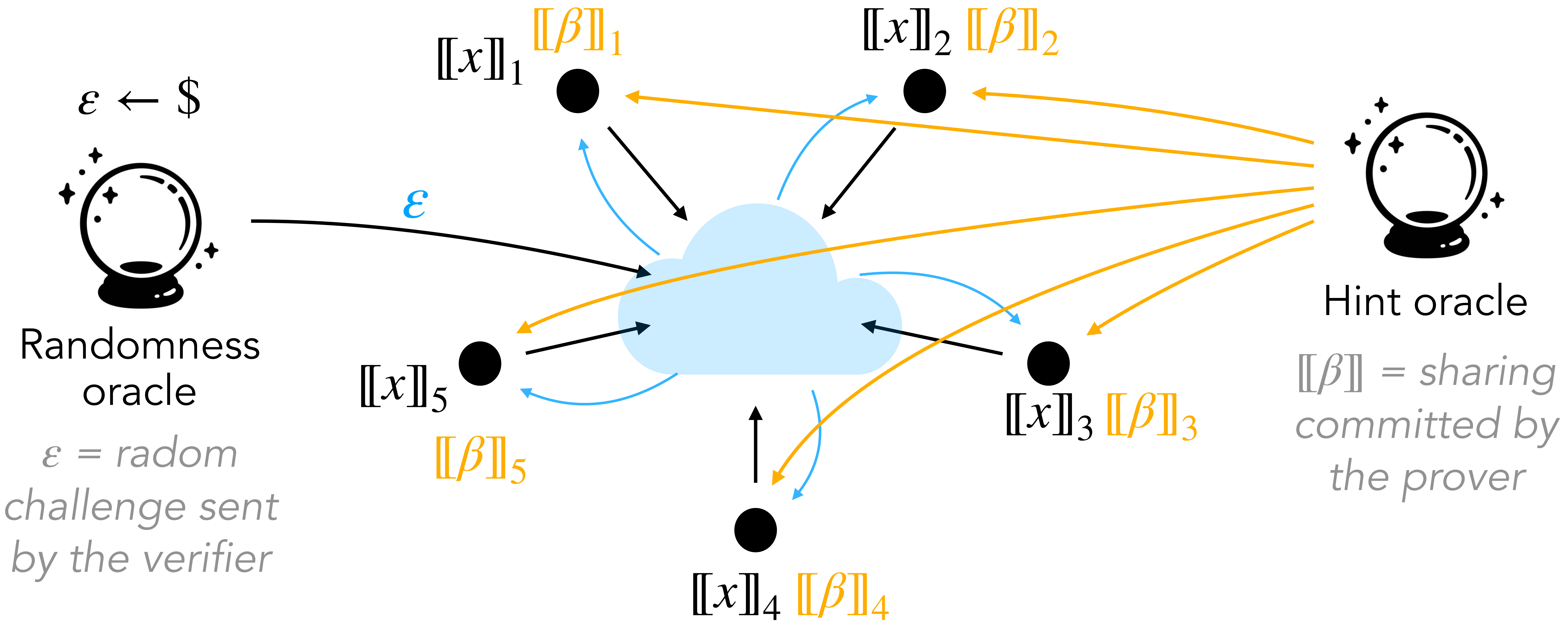

False positive probability

- False positive = MPC protocol outputs "Accept" while $[\![x]\!]$ s.t. $F(x) \neq y$

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- False positive probability:

$$p = \max_{[\![\beta]\!]} P\left[\text{MPC} : ([\![x]\!], [\![\beta]\!], \varepsilon) \mapsto \text{"Accept"} \mid F(x) \neq y\right]$$

(over the randomness of ε)

False positive probability

- False positive = MPC protocol outputs “Accept” while $[\![x]\!]$ s.t. $F(x) \neq y$
- False positive probability:

$$p \;=\; \max_{[\![\beta]\!]} P\left[\text{MPC} : ([\![x]\!], [\![\beta]\!], \varepsilon) \mapsto \text{"Accept"} \mid F(x) \neq y\right]$$

(over the randomness of ε)

- Soundness error:

$$\boxed{\frac{1}{N}} \longrightarrow \boxed{\frac{1}{N} + p}$$

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$\cdots$

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$

$\cdots$

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$

$\leftarrow$ **hint** $ab = c$

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$

ε

$\cdots$

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$

ε

$\leftarrow$ ***hint*** $ab = c$

$\leftarrow$ ***random*** ε

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$

ε

$[\![\alpha]\!]_1 = \varepsilon[\![x]\!]_1 + [\![a]\!]_1$

$[\![\beta]\!]_1 = [\![y]\!]_1 + [\![b]\!]_1$

$\downarrow$

$[\![\alpha]\!]_1 \ [\![\beta]\!]_1$

…

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$

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$[\![\alpha]\!]_N = \varepsilon[\![x]\!]_N + [\![a]\!]_N$

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$[\![\alpha]\!]_N \ [\![\beta]\!]_N$

← **hint** $ab = c$

← **random** ε

$\alpha = \epsilon x + a$

$\beta = y + b$

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$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$

ε

$[\![\alpha]\!]_1 = \varepsilon[\![x]\!]_1 + [\![a]\!]_1$

$[\![\beta]\!]_1 = [\![y]\!]_1 + [\![b]\!]_1$

$[\![\alpha]\!]_1 \ [\![\beta]\!]_1$

$[\![v]\!]_1 = \epsilon[\![z]\!]_1 - [\![c]\!]_1 + \alpha[\![b]\!]_1 + \beta[\![a]\!]_1 - \alpha\beta$

$[\![v]\!]_1$

$\cdots$

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$

ε

$[\![\alpha]\!]_N = \varepsilon[\![x]\!]_N + [\![a]\!]_N$

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$[\![\alpha]\!]_N \ [\![\beta]\!]_N$

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← **hint** $ab = c$

← **random** ε

$\alpha = \epsilon x + a$

$\beta = y + b$

v

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$

$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$

ε

$[\![\alpha]\!]_1 = \varepsilon[\![x]\!]_1 + [\![a]\!]_1$

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$[\![v]\!]_1$

…

$\mathscr{P}_N$

$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$

$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$

ε

$[\![\alpha]\!]_N = \varepsilon[\![x]\!]_N + [\![a]\!]_N$

$[\![\beta]\!]_N = [\![y]\!]_N + [\![b]\!]_N$

$[\![\alpha]\!]_N \ [\![\beta]\!]_N$

$[\![v]\!]_N = \epsilon[\![z]\!]_N - [\![c]\!]_N + \alpha[\![b]\!]_N + \beta[\![a]\!]_N - \alpha\beta$

$[\![v]\!]_N$

← hint $ab = c$

← random ε

$\alpha = \epsilon x + a$

$\beta = y + b$

v

$$g(v) = \begin{cases} \text{Accept} & \text{if } v = 0 \\ \text{Reject} & \text{if } v \neq 0 \end{cases}$$

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$$
$$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$$
$$\varepsilon$$
$$[\![\alpha]\!]_1 = \varepsilon[\![x]\!]_1 + [\![a]\!]_1$$
$$[\![\beta]\!]_1 = [\![y]\!]_1 + [\![b]\!]_1$$

$$\downarrow$$

$$[\![\alpha]\!]_1 \; [\![\beta]\!]_1$$

$$[\![v]\!]_1 = \epsilon[\![z]\!]_1 - [\![c]\!]_1 + \alpha[\![b]\!]_1 + \beta[\![a]\!]_1 - \alpha\beta$$

$$\downarrow$$

$$[\![v]\!]_1$$

$\cdots$

$\mathscr{P}_N$

$$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$$
$$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$$
$$\varepsilon$$
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$$\downarrow$$

$$[\![\alpha]\!]_N \; [\![\beta]\!]_N$$

$$[\![v]\!]_N = \epsilon[\![z]\!]_N - [\![c]\!]_N + \alpha[\![b]\!]_N + \beta[\![a]\!]_N - \alpha\beta$$

$$\downarrow$$

$$[\![v]\!]_N$$

← **hint** $ab = c$

← **random** ε

$$\alpha = \epsilon x + a$$
$$\beta = y + b$$

$$v$$

$$g(v) = \begin{cases} \text{Accept} & \text{if } v = 0 \\ \text{Reject} & \text{if } v \neq 0 \end{cases}$$

If $xy = z$ and $ab = c$, then $v = 0$

If $xy \neq z$ or $ab \neq c$, then $\Pr[v = 0] = 1/|\mathbb{F}|$

Example: [BN20] check product $xy = z$

$\mathscr{P}_1$

$$[\![x]\!]_1, [\![y]\!]_1, [\![z]\!]_1$$
$$[\![a]\!]_1, [\![b]\!]_1, [\![c]\!]_1$$
$$\varepsilon$$
$$[\![\alpha]\!]_1 = \varepsilon[\![x]\!]_1 + [\![a]\!]_1$$
$$[\![\beta]\!]_1 = [\![y]\!]_1 + [\![b]\!]_1$$

$\downarrow$ $[\![\alpha]\!]_1 \; [\![\beta]\!]_1$

$$[\![v]\!]_1 = \epsilon[\![z]\!]_1 - [\![c]\!]_1 + \alpha[\![b]\!]_1 + \beta[\![a]\!]_1 - \alpha\beta$$

$\downarrow$ $[\![v]\!]_1$

…

$\mathscr{P}_N$

$$[\![x]\!]_N, [\![y]\!]_N, [\![z]\!]_N$$
$$[\![a]\!]_N, [\![b]\!]_N, [\![c]\!]_N$$
$$\varepsilon$$
$$[\![\alpha]\!]_N = \varepsilon[\![x]\!]_N + [\![a]\!]_N$$
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$$[\![v]\!]_N = \epsilon[\![z]\!]_N - [\![c]\!]_N + \alpha[\![b]\!]_N + \beta[\![a]\!]_N - \alpha\beta$$

$\downarrow$ $[\![v]\!]_N$

← **hint** $ab = c$

← **random** ε

$\alpha = \epsilon x + a$
$\beta = y + b$

false positive probability

$$g(v) = \begin{cases} \text{Accept} & \text{if } v = 0 \\ \text{Reject} & \text{if } v \neq 0 \end{cases}$$

If $xy = z$ and $ab = c$, then $v = 0$

If $xy \neq z$ or $ab \neq c$, then $\Pr[v = 0] = 1/|\mathbb{F}|$

Verifying arbitrary circuits

- Product-check protocol ⇒ protocol for checking any arithmetic circuit $C(x) = y$
- Principle:
  - Let $\{c_i = a_i \cdot b_i\}$ all the multiplications in C
  - Extended witness: $w = x \,\|\, (c_1, \ldots, c_m)$
  - Compute $[\![y]\!]$ = linear function of $[\![w]\!]$ → check $[\![y]\!]$ = sharing of y
  - $[\![a_i]\!], [\![b_i]\!], [\![c_i]\!]$ = linear functions of $[\![w]\!]$ → product check on $[\![a_i]\!], [\![b_i]\!], [\![c_i]\!]$

Optimisations

Optimising communication (sig. size)

- **Signature = transcript P → V** (×τ iterations)
  - $\{\mathrm{Com}^{\rho_i}([\![x]\!]_i)\}$ → N commitments
  - $[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$ → N MPC broadcasts
  - $\{[\![x]\!]_i, \rho_i\}_{i \neq i^*}$ → $N - 1$ input shares + random tapes

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- First optimisation: **hashing**
  - $[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N \rightarrow h = \mathrm{Hash}([\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N), \ \alpha = \Sigma_i [\![\alpha]\!]_i$
  - Verification
    - $[\![\alpha]\!]_i = \varphi([\![x]\!]_i) \quad \forall i \neq i*$
    - $[\![\alpha]\!]_{i*} = \alpha - \Sigma_{i \neq i*} [\![\alpha]\!]_i$
    - Check $\mathrm{Hash}([\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N) = h$

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  - $\{[\![x]\!]_i, \rho_i\}_{i \neq i^*}$ → $N-1$ input shares + random tapes **main cost**
- First optimisation: **hashing**
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Second optimisation: seed trees

- **[KKW18]** Katz, Kolesnikov, Wang: “Improved Non-Interactive Zero Knowledge with Applications to Post-Quantum Signatures” (CCS 2018)

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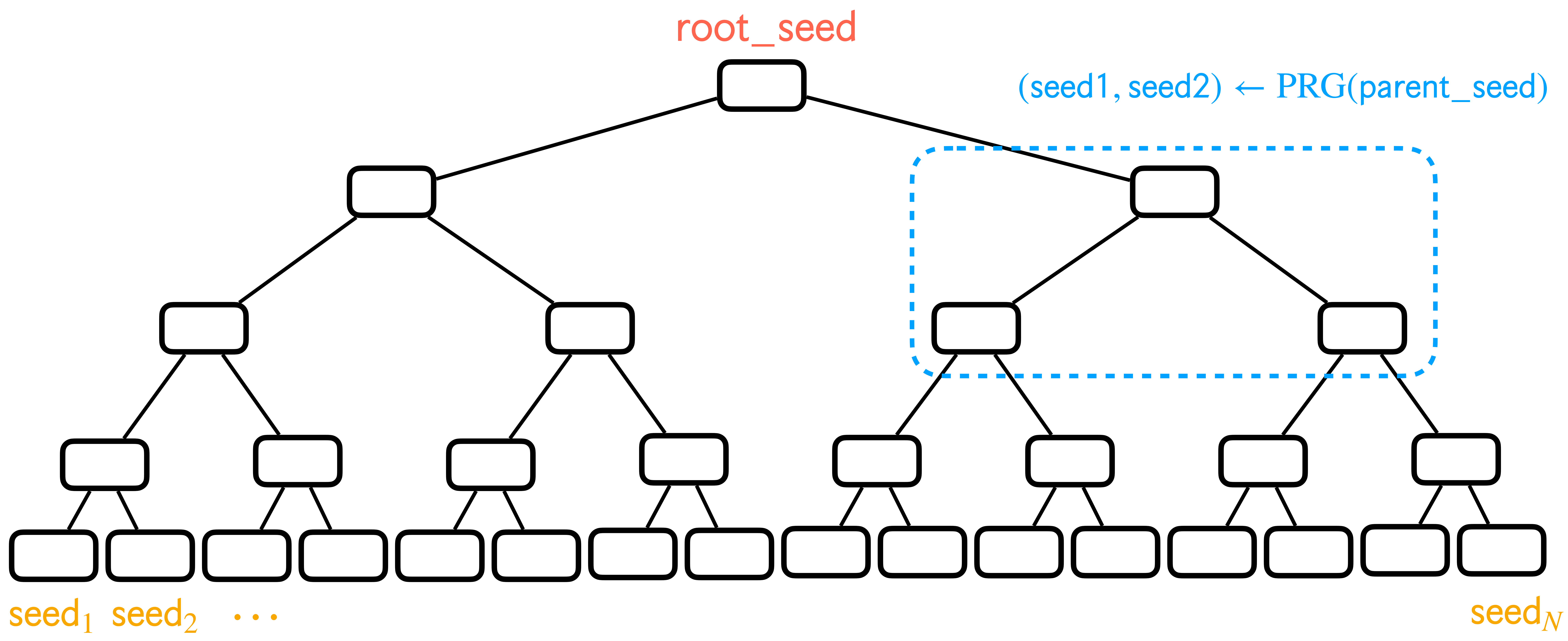
- Seeds $\{\mathsf{seed}_i\}$ generated from a common "root seed"
- Goal: revealing $\{\mathsf{seed}_i\}_{i \neq i*}$ with less than $(N-1) \cdot \lambda$ bits

Second optimisation: seed trees



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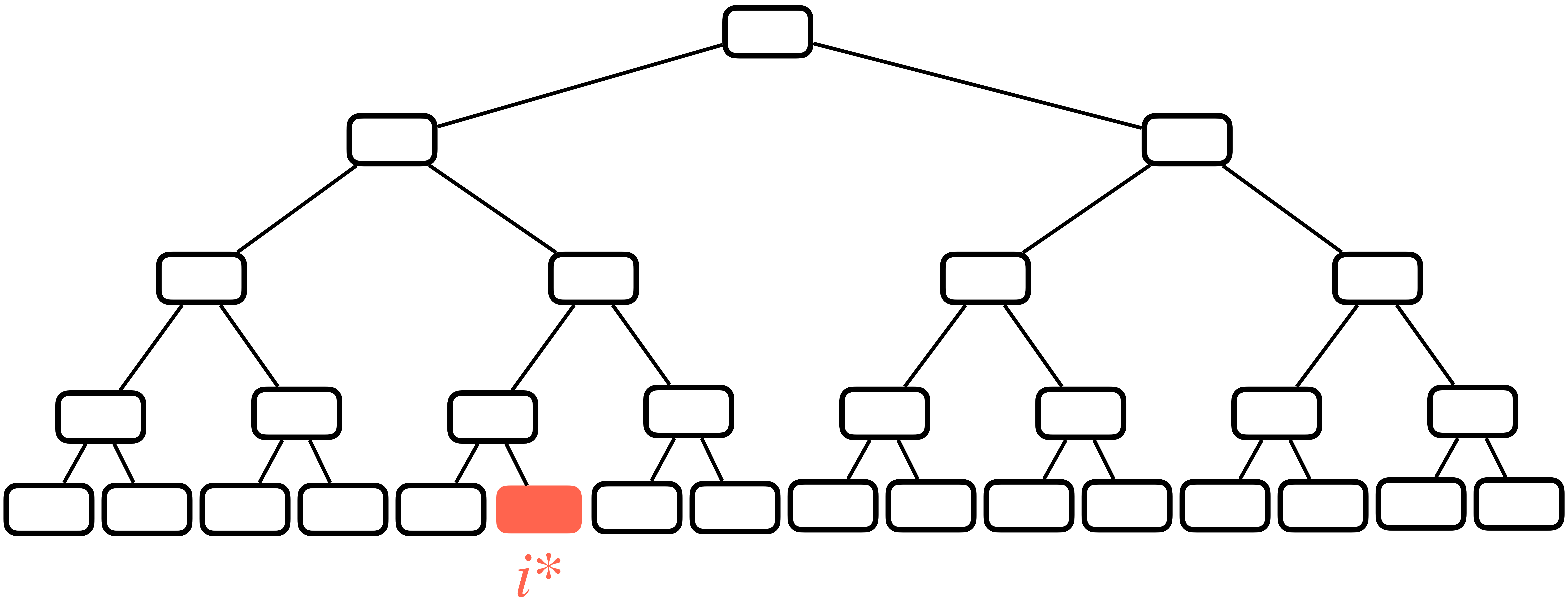

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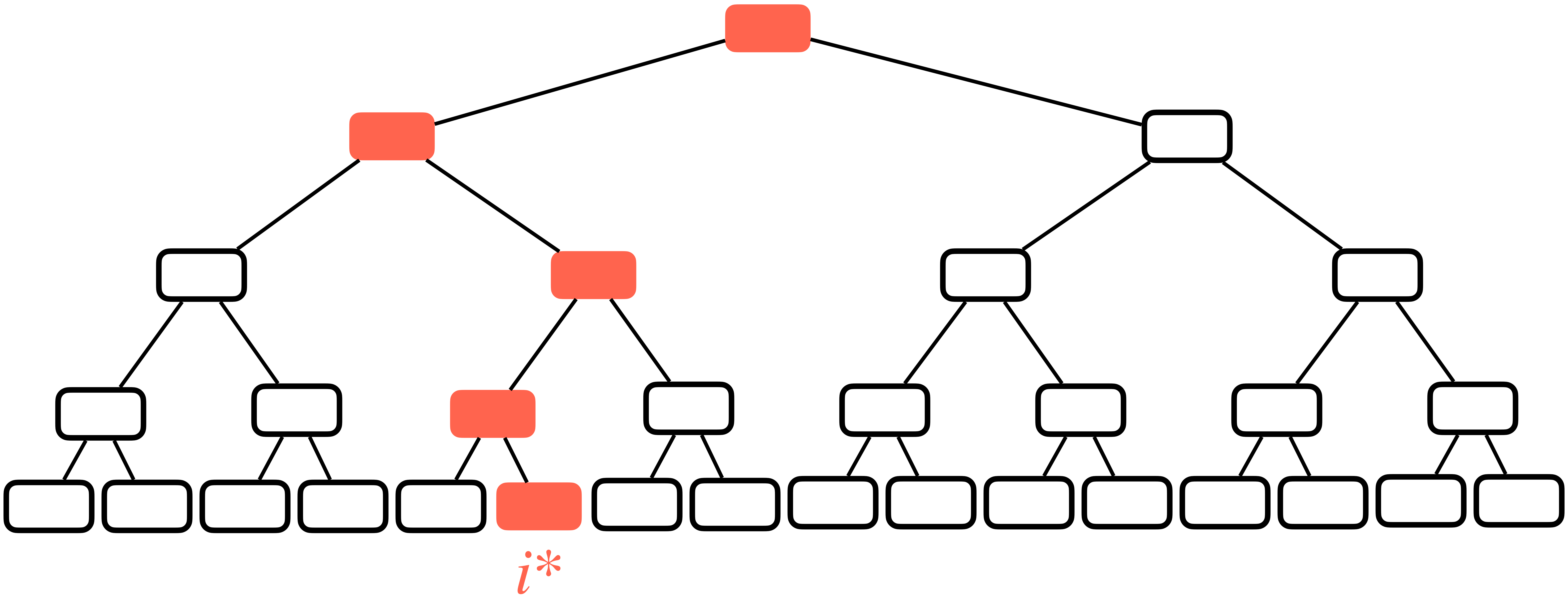

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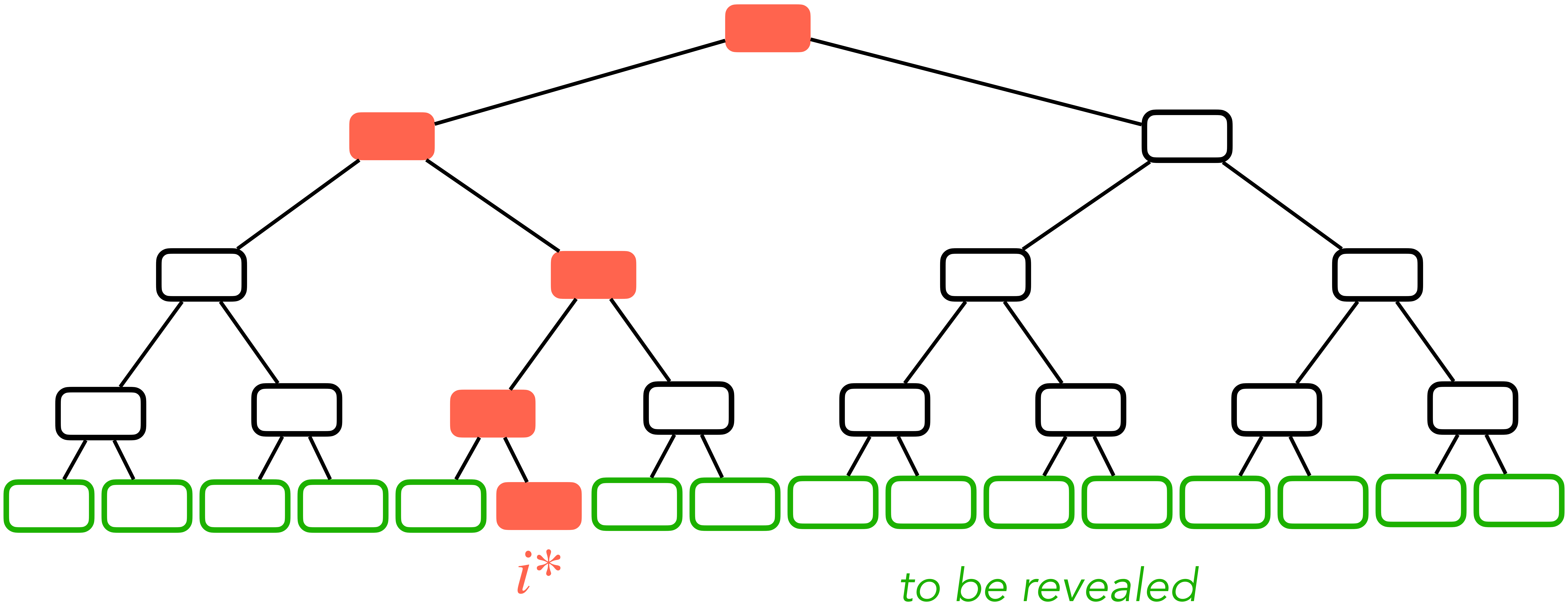

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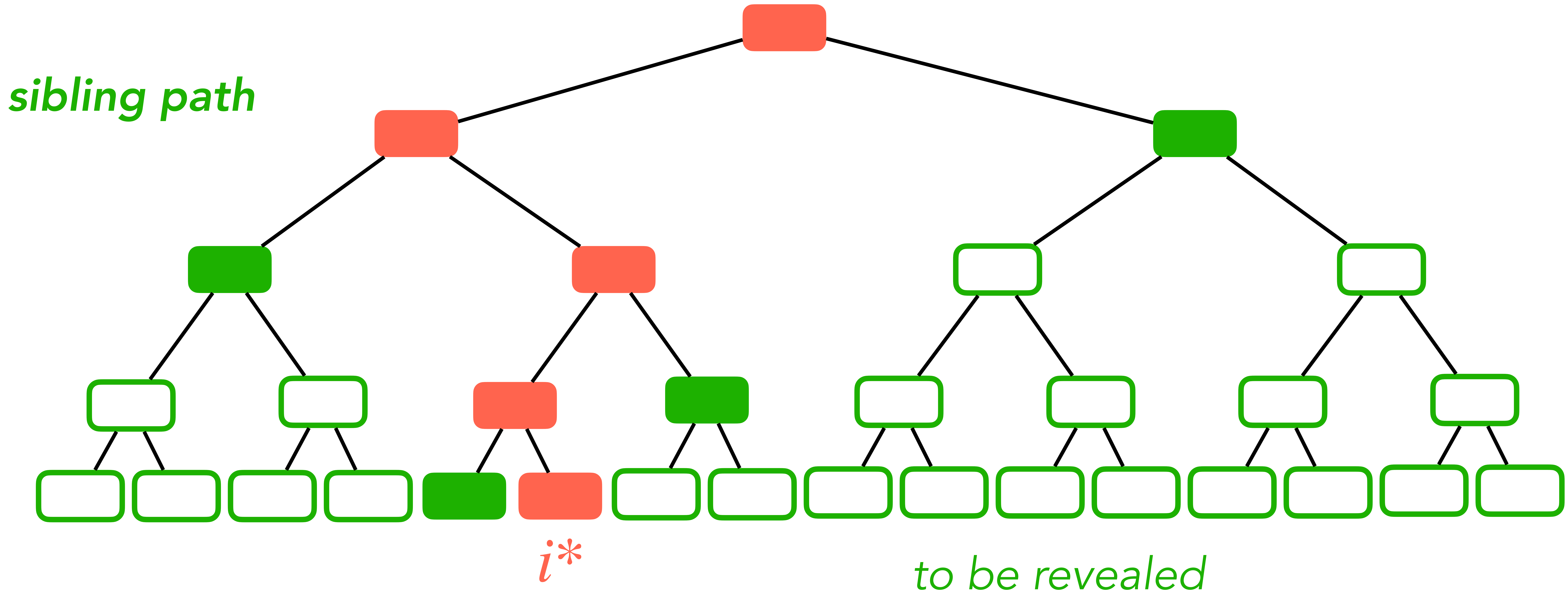

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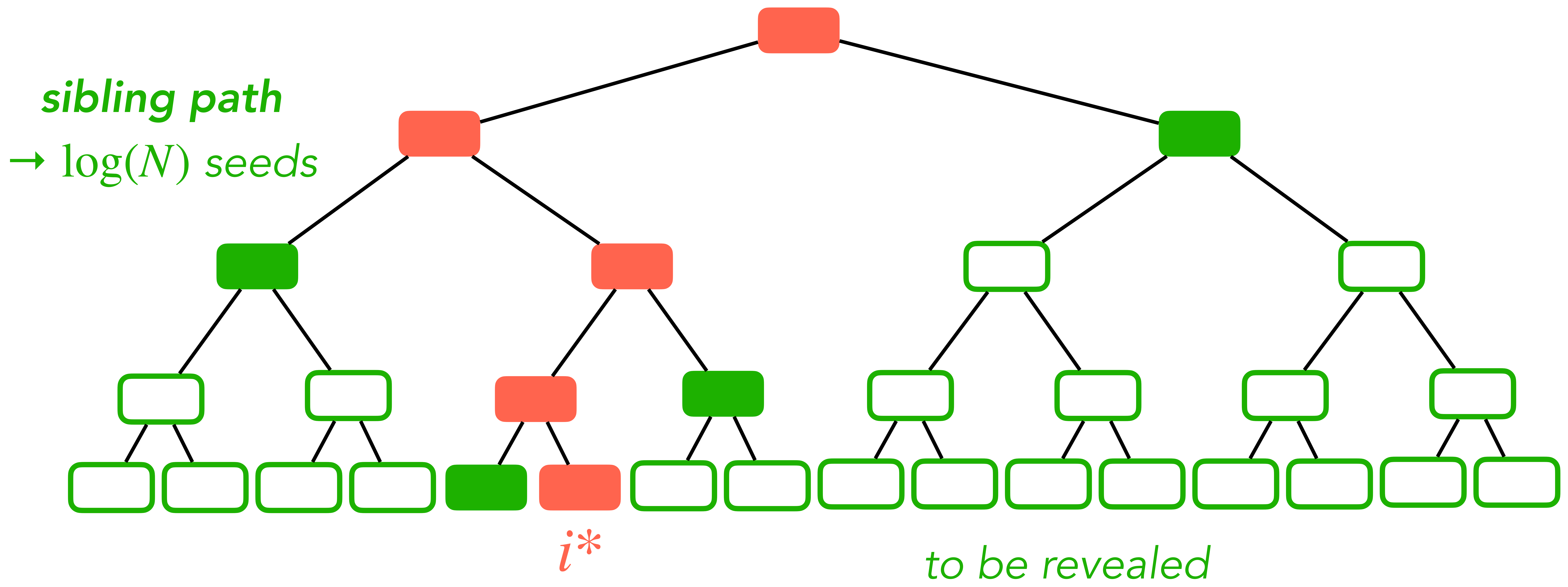

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  - $\{[\![x]\!]_i, \rho_i\}_{i \neq i^*}$ → ~~$N-1$ input shares + random tapes~~ → $\log(N)$ seeds

    \+ $[\![x]\!]_N$ if $i^* \neq N$

Second optimisation: seed trees

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  - $\{[\![x]\!]_i, \rho_i\}_{i \neq i^*}$ → ~~$N-1$ input shares + random tapes~~ → $\log(N)$ seeds + $[\![x]\!]_N$ if $i^* \neq N$
- Verification
  - Sibling path → $\{\text{seed}_i\}_{i \neq i^*}$
  - $\text{seed}_i \to ([\![x]\!]_i, \rho_i) \quad \forall\, i \neq i^*$
  - …

Optimising computation: hypercube technique

- **[AGHHJY23]** Aguilar Melchor, Gama, Howe, Hülsing, Joseph, Yue. "The Return of the SDitH" (EUROCRYPT 2023)

Optimising computation: hypercube technique

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- High-level principle

  - Apply MPC computation to sums of shares

$$\Sigma_{i \in I} [\![x_i]\!] \xrightarrow{\varphi} \Sigma_{i \in I} [\![\alpha_i]\!]$$

  - Only $\log N + 1$ such party computations necessary for the prover

  - Only $\log N$ for the verifier

Optimising computation: hypercube technique

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  - Only $\log N + 1$ such party computations necessary for the prover
  - Only $\log N$ for the verifier
- See Nicolas Gama's talk at EC: https://youtu.be/z6nE4fOWvZA (49:33)

SDitH Signature Scheme: MPCitH with SD

Syndrome decoding problem

- Parameters
  - A field $\mathbb{F}_q$, $m \in \mathbb{N}$ *(code length)* , $k < m$ *(code dimension)* , $w < m$ *(weight)*
- Let
  - $H \leftarrow \mathbb{F}_q^{(m-k)\times m}$ *(random parity-check matrix)*
  - $x \leftarrow \mathbb{F}_q^m$ s.t. $\mathrm{wt}(x) \leq w$ *(SD solution)*
  - $y = Hx$ *(syndrome)*
- From (H, y) find x

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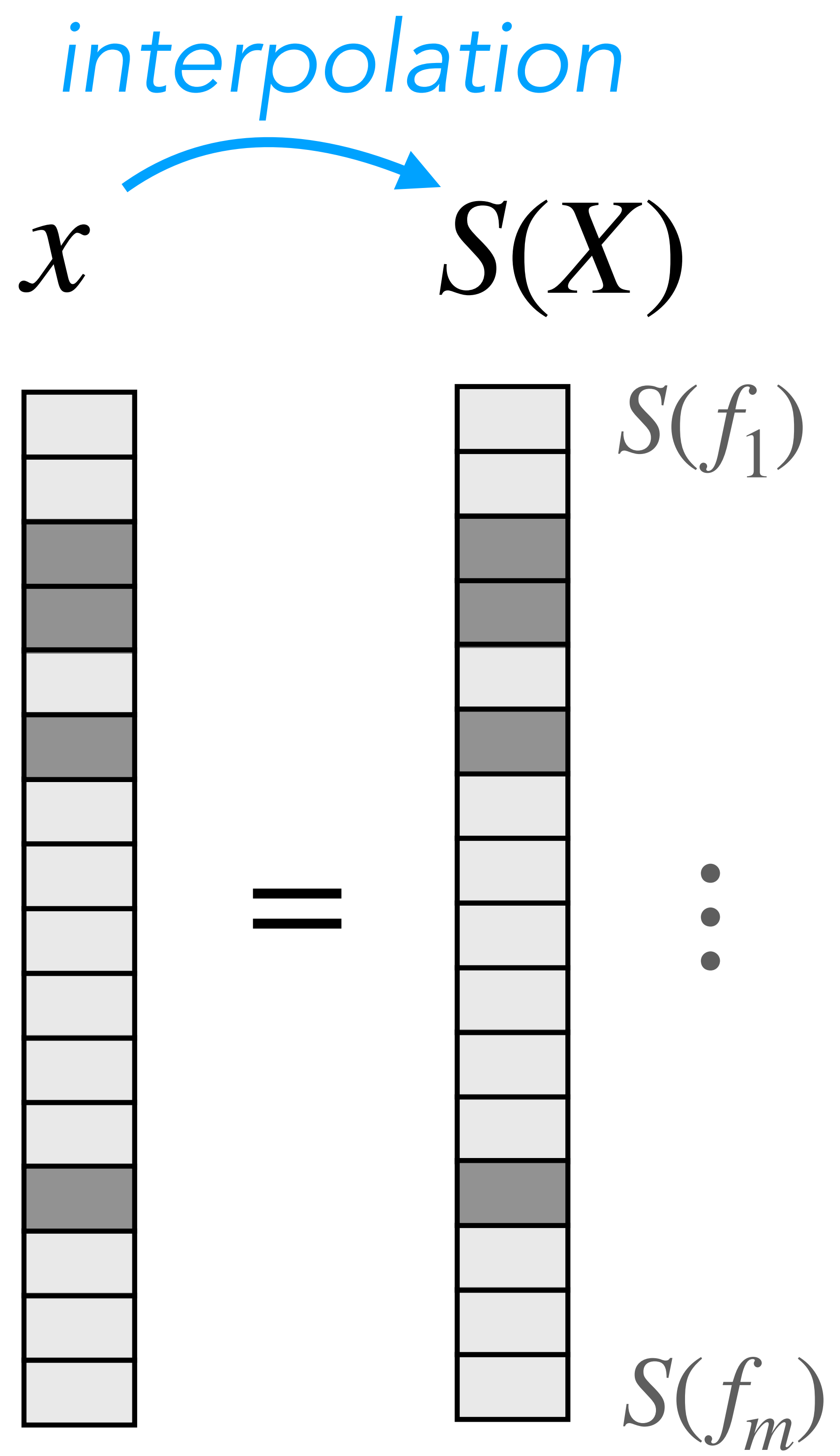
- From (H, y) find x
- Standard form (wlog): $H = (H' \,|\, I_{m-k})$ $\Rightarrow$ $y = H'x_A + x_B$ where $x = (\underbrace{x_A}_{|x_A| = k} \,|\, \underbrace{x_B}_{|x_B| = m-k})$

$\Rightarrow$ $x_B = y - H'x_A$

Polynomial expression



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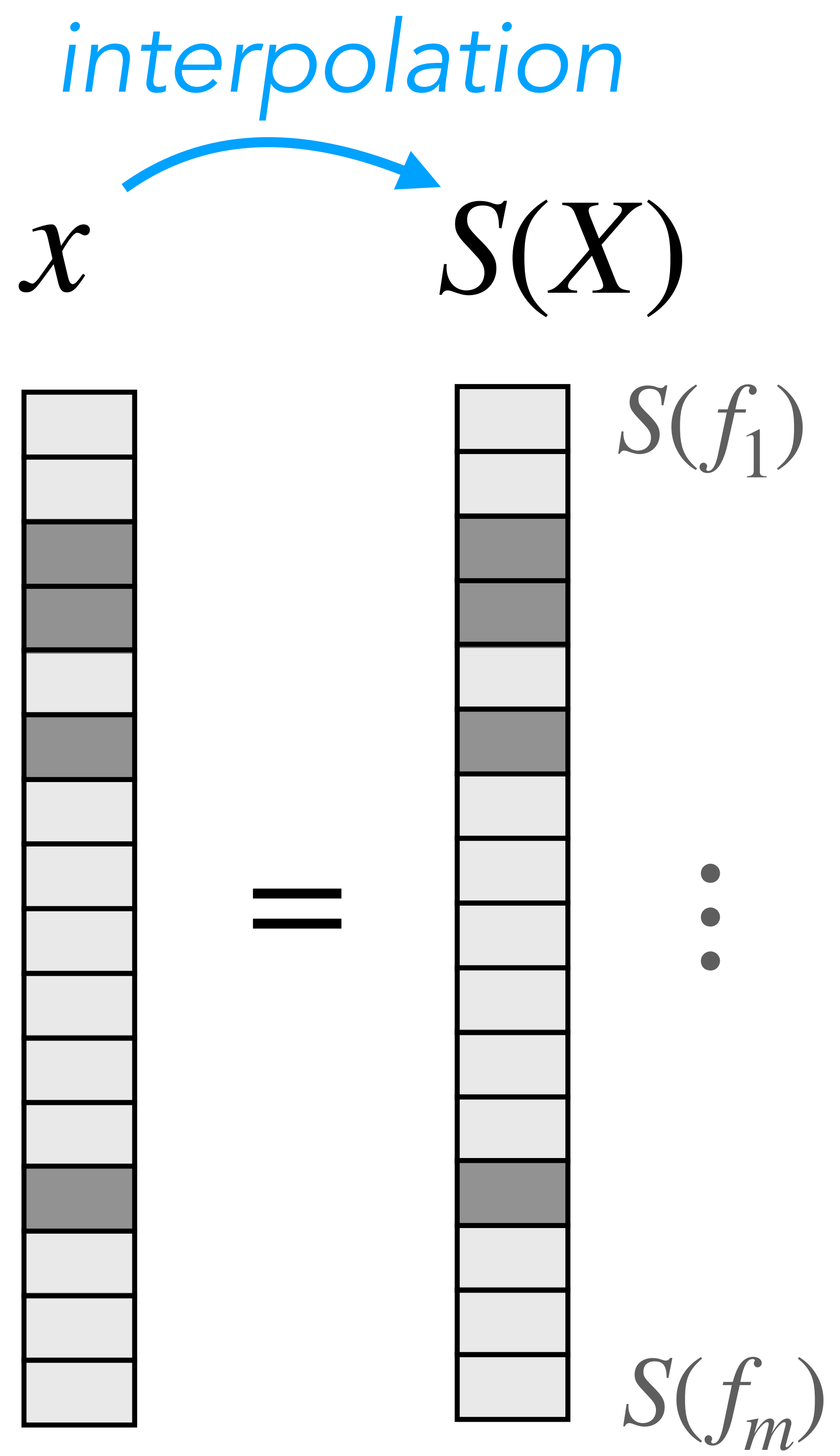
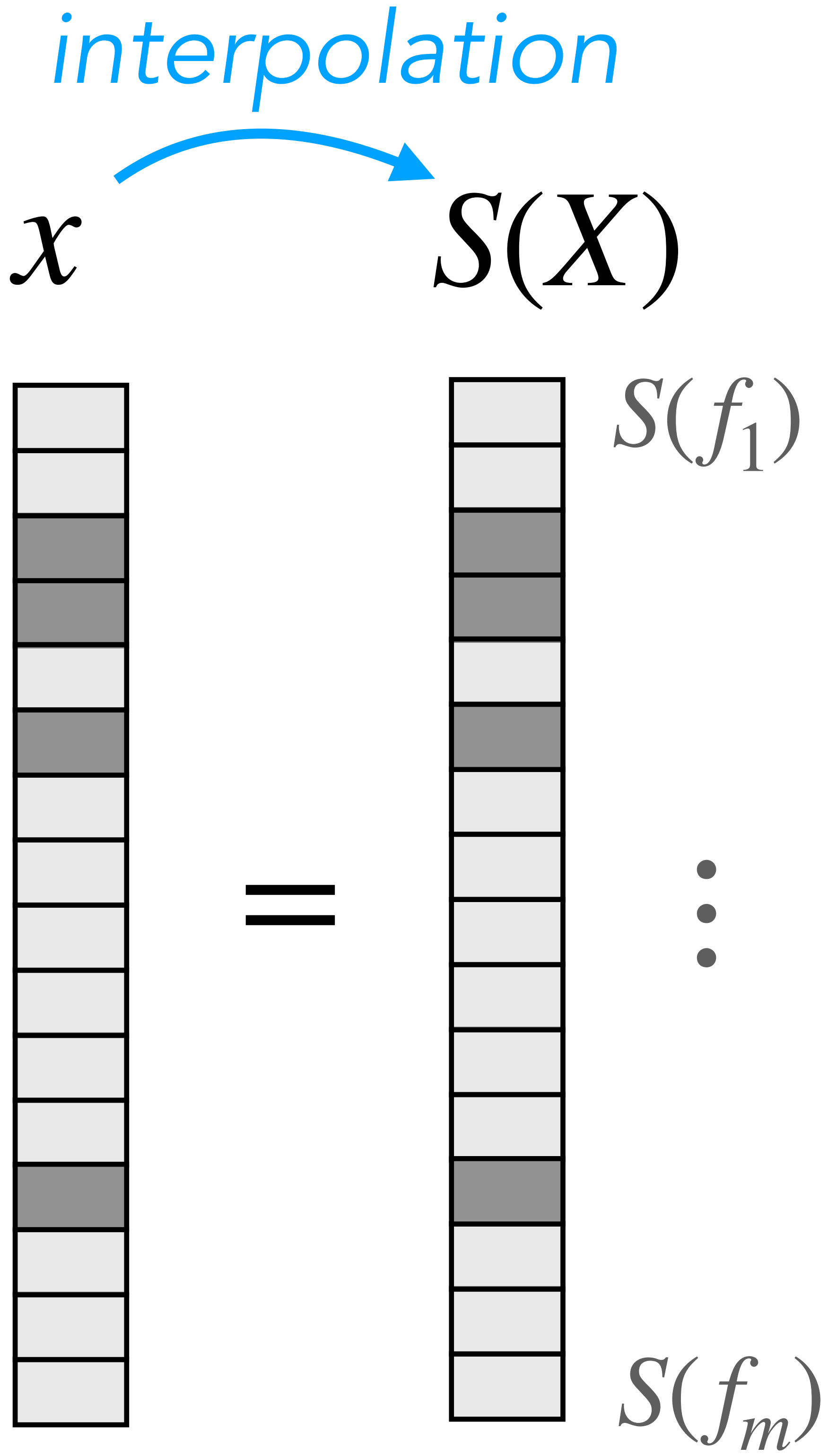


$$Q(X) = \prod_{i \in E} (X - f_i)$$

Polynomial expression

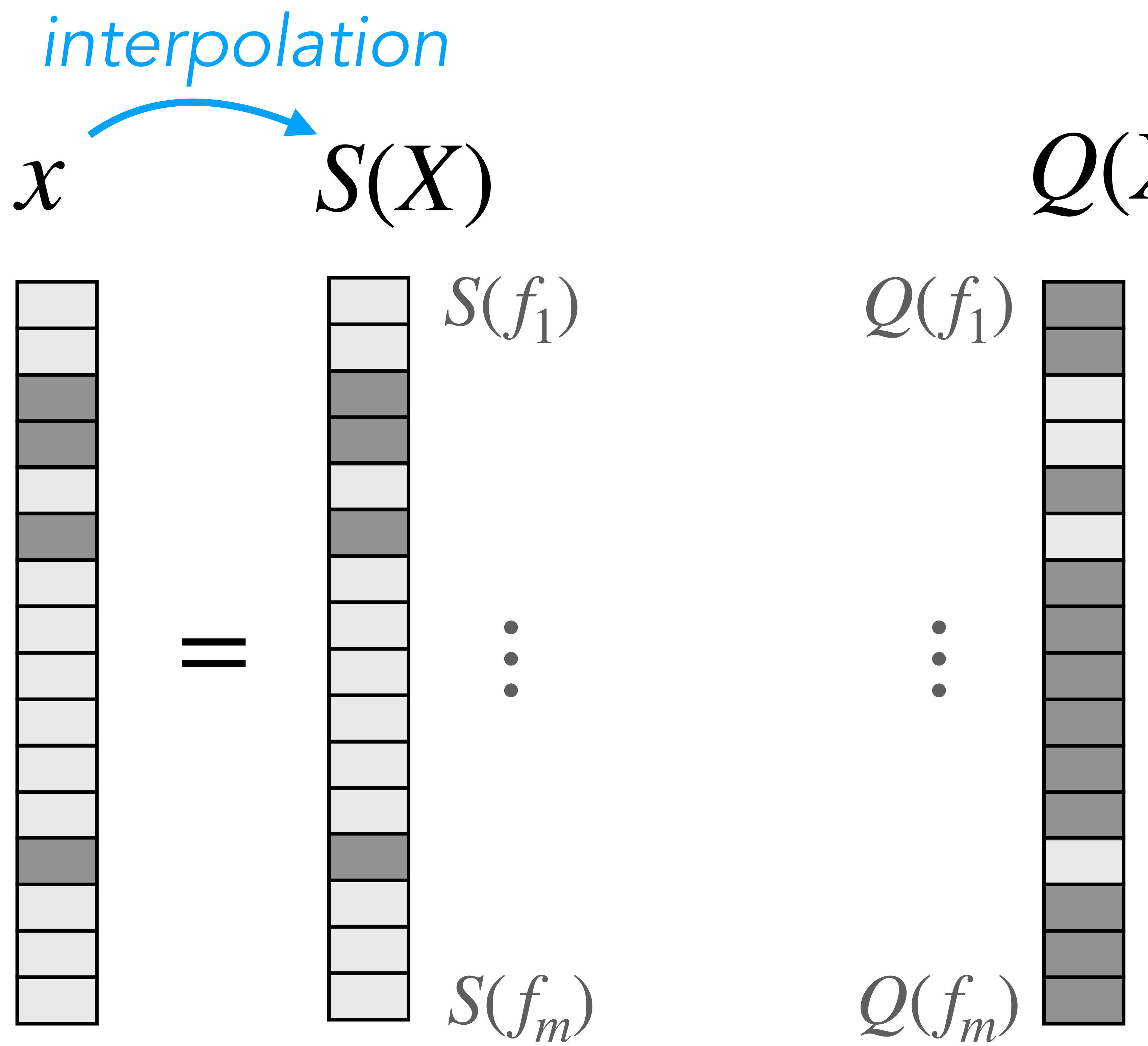


$$Q(X) = \prod_{i \in E} (X - f_i)$$

indices i s.t. $x_i \neq 0$

$|E| \leq w \;\Rightarrow\; \deg(Q) \leq w$

Polynomial expression



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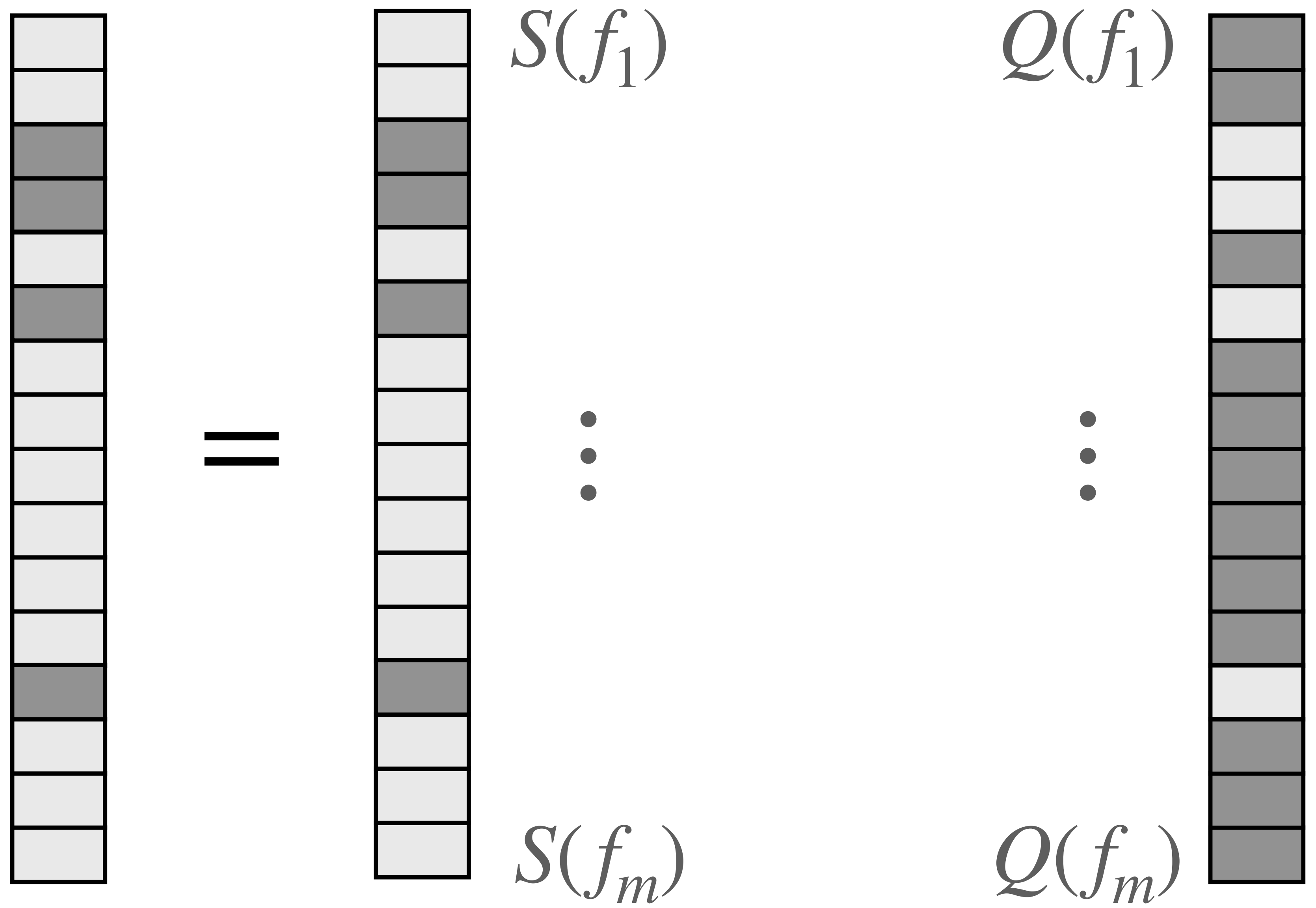
= zero coordinate

= non-zero coordinate

Polynomial expression

$$x \xrightarrow{\text{interpolation}} S(X) \qquad\qquad Q(X) = \prod_{i \in E} (X - f_i)$$



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= zero coordinate

= non-zero coordinate

$\Rightarrow\; S(X) \cdot Q(X)$ evaluates to 0 in $f_1, \ldots, f_m$

some degree $\leqslant w - 1$ *polynomial*

$$\Rightarrow\; S(X) \cdot Q(X) = \underbrace{F(X)}_{\prod_{i \in [1:m]} (X - f_i)} \cdot P(X)$$

Polynomial expression

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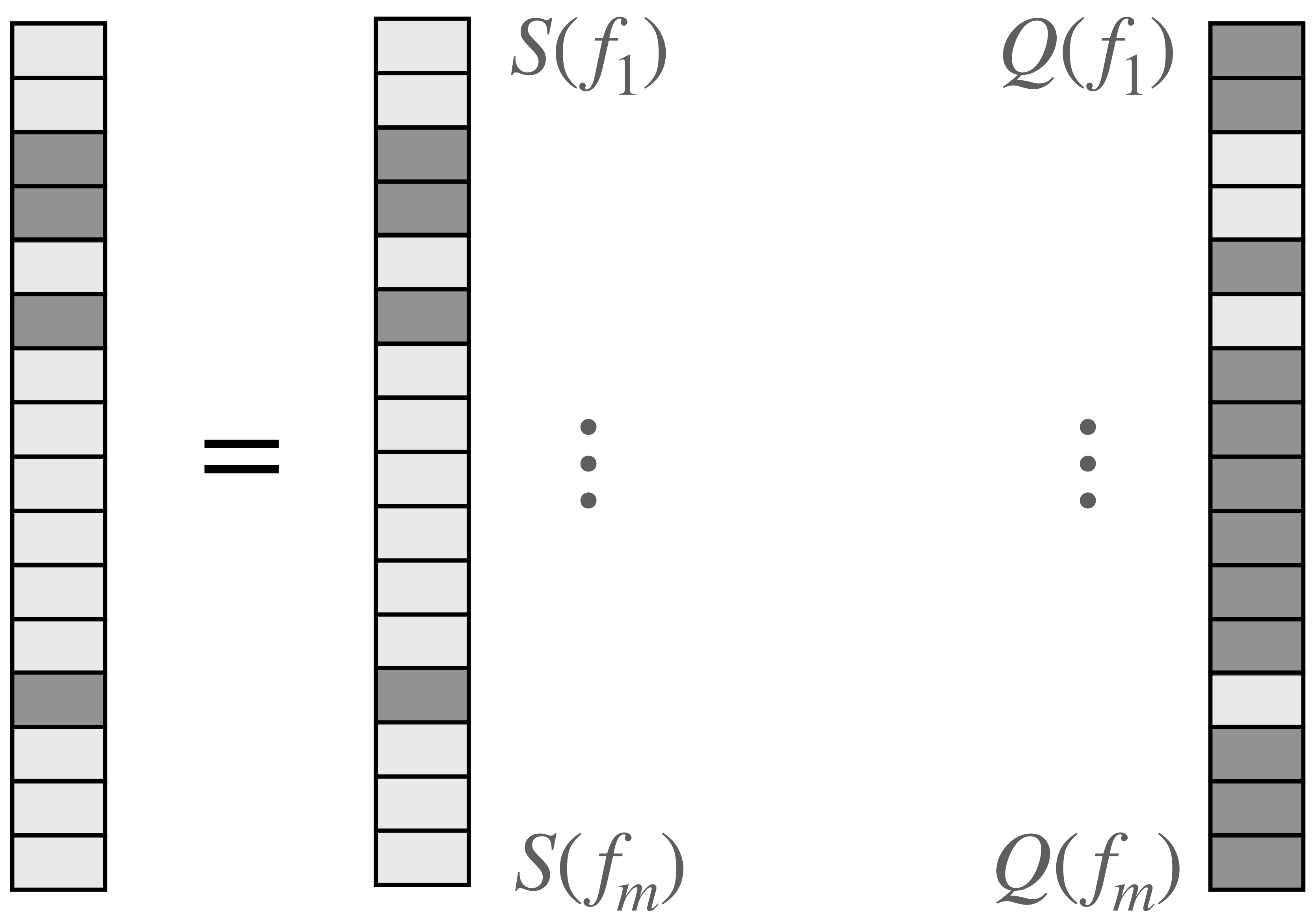

$\Rightarrow$ $S(X) \cdot Q(X)$ evaluates to 0 in $f_1, \ldots, f_m$

If $\mathrm{wt}(x) \leq w$ then

$\exists\, Q$ of degree $\leq w$ s.t. $S(X) \cdot Q(X)$

evaluates to 0 in $f_1, \ldots, f_m$

$\Leftrightarrow$

$\exists\, Q, P$ of degrees $\leq w$, $w - 1$ s.t

$S(X) \cdot Q(X) = F(X) \cdot P(X)$

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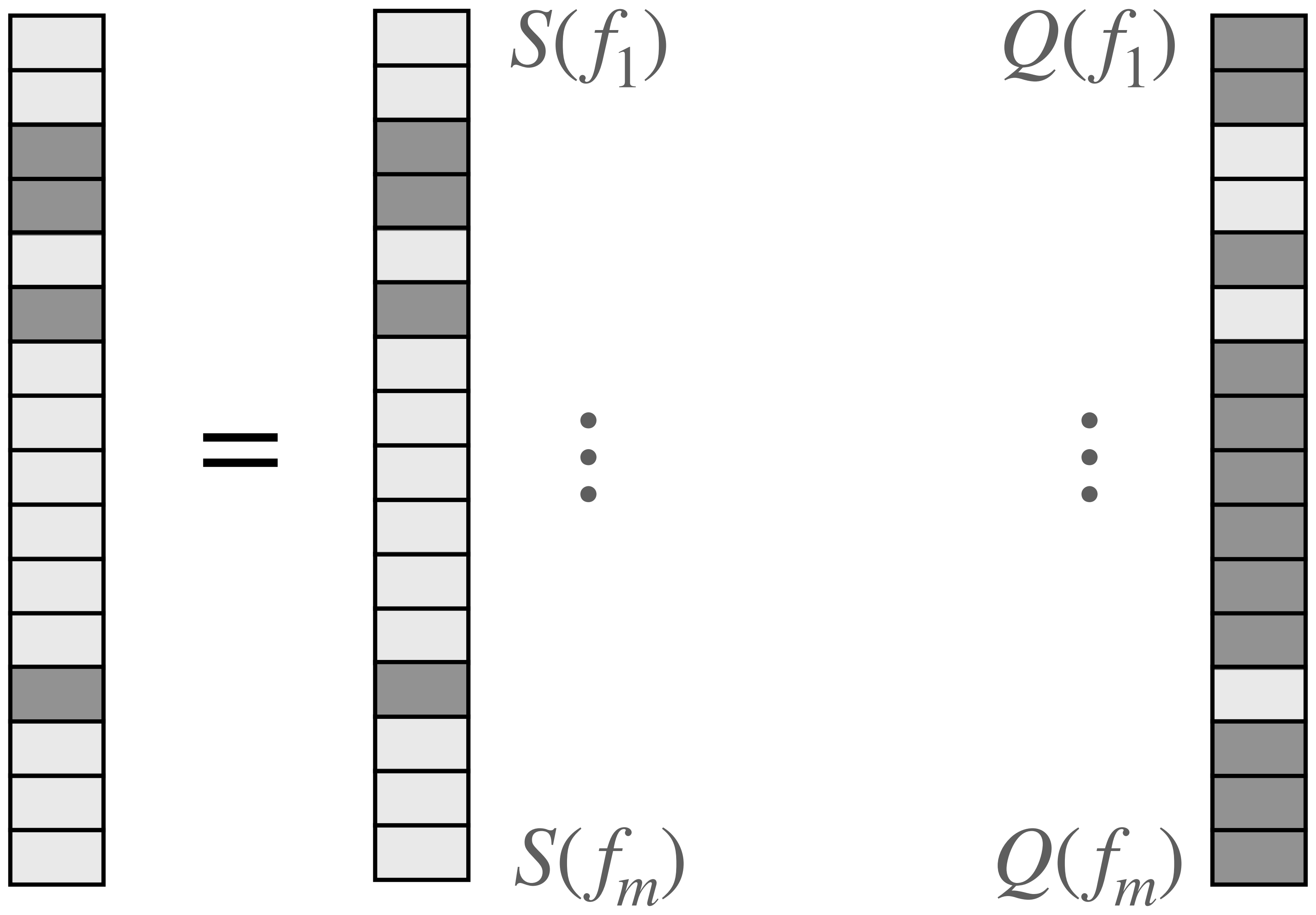

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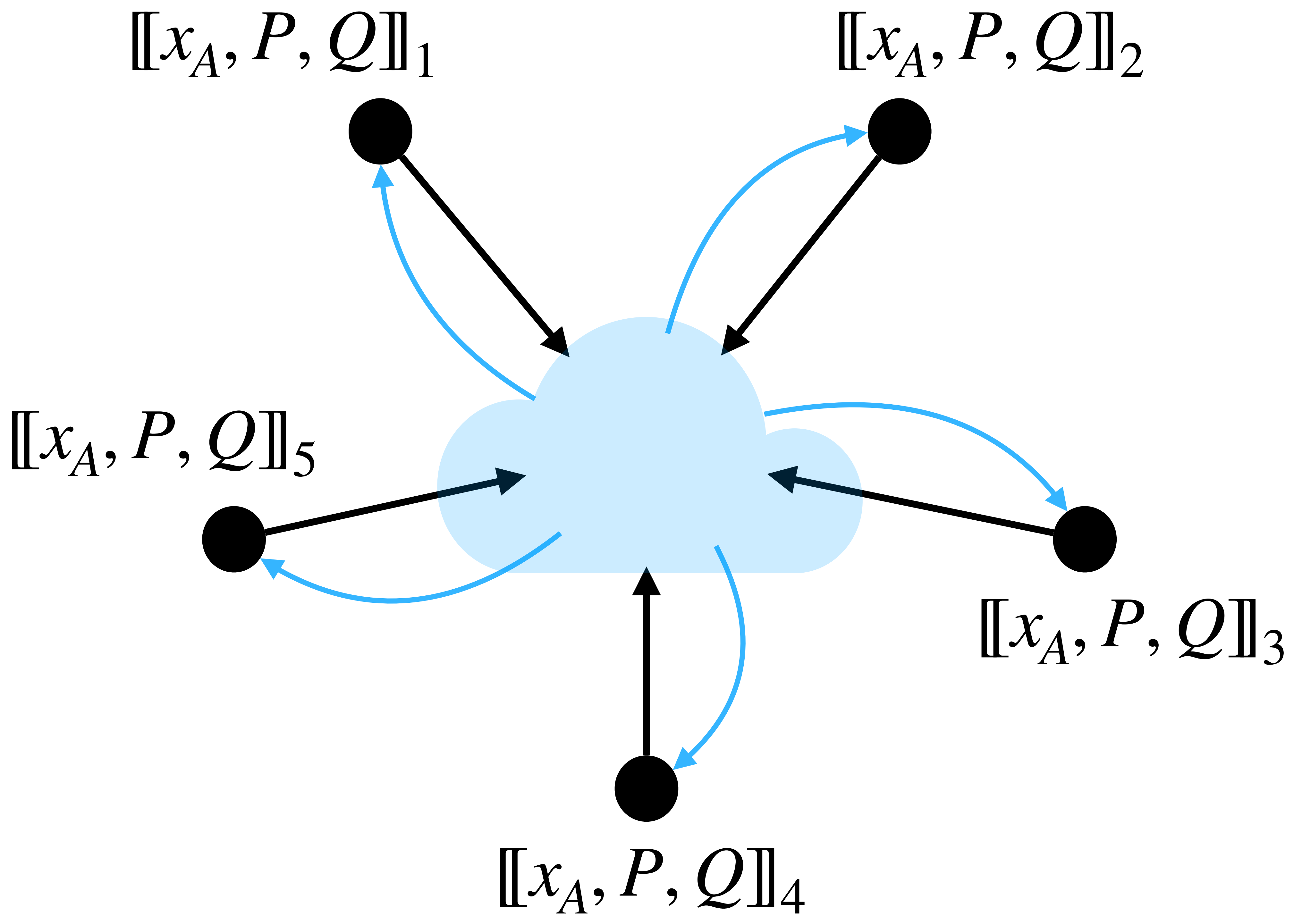
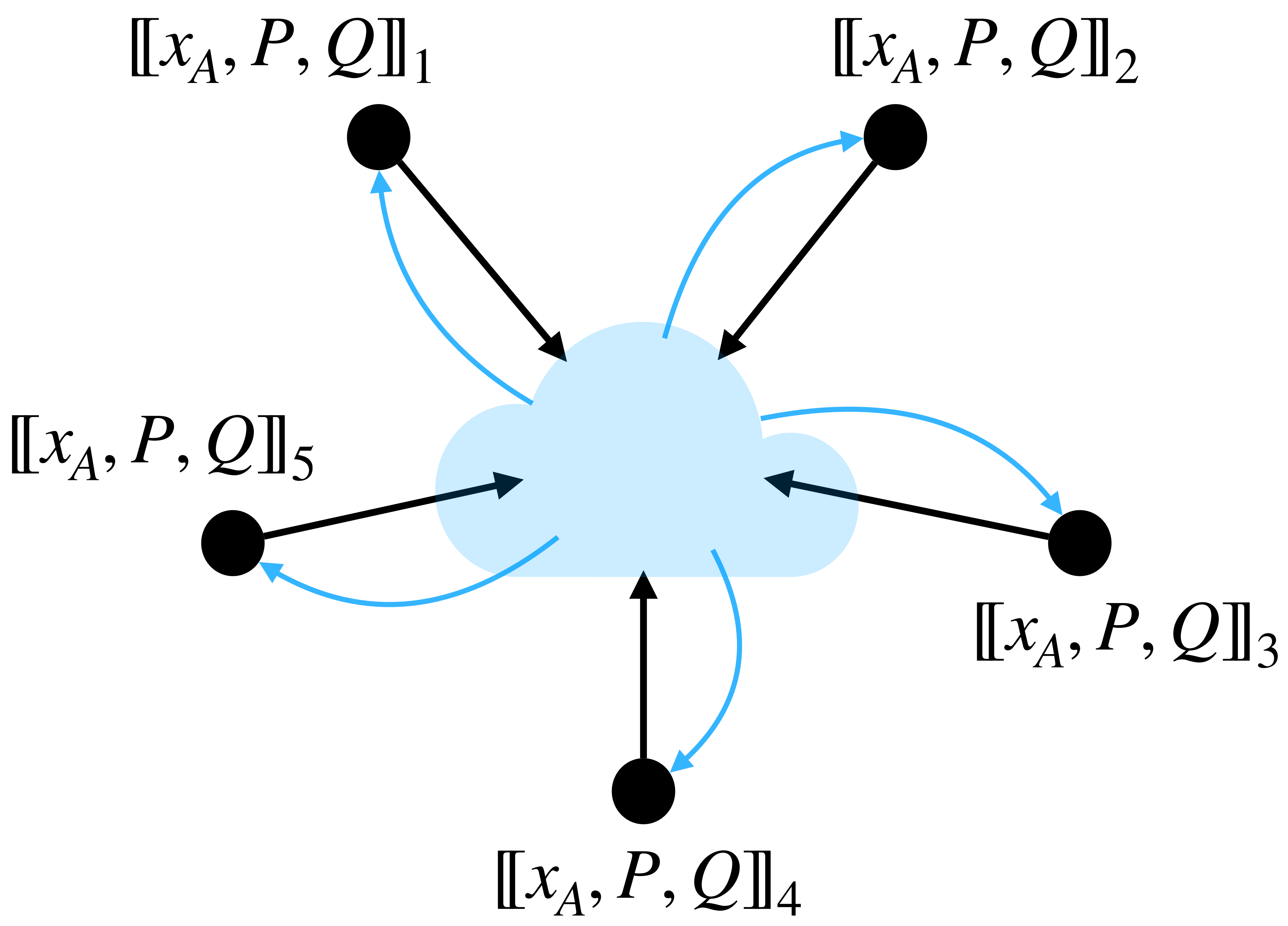
We'll show this

SDitH MPC protocol



- **Parties receive**
  - $[\![x_A]\!], [\![P]\!], [\![Q]\!]$ sharings of x_A, P, Q
  - (H', y) SD instance

SDitH MPC protocol



- **Parties receive**
  - $[\![x_A]\!], [\![P]\!], [\![Q]\!]$ sharings of x_A, P, Q
  - (H', y) SD instance
- **Parties jointly compute**

$$g(x_A, P, Q) = \begin{cases} \text{Accept} & \text{if } SQ = FP \\ \text{Reject} & \text{otherwise} \end{cases}$$

where $x_B = y - H'x_A$ and $S = \mathrm{Interp}(x_A \mid x_B)$

Schwartz–Zippel lemma

- Let P_1 and P_2 two degree-d polynomials of $\mathbb{F}[X]$
- Let r a random point of $\mathbb{F}$,

$$\Pr\left[P_1(r) = P_2(r) \mid P_1 \neq P_2\right] \leq \frac{d}{|\mathbb{F}|}$$

($P_1(r) = P_2(r) \quad \Leftrightarrow \quad r \in$ roots of $P_1 - P_2$)

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$$(\, P_1(r) = P_2(r) \quad \Leftrightarrow \quad r \in \text{roots of } P_1 - P_2 \,)$$

- For a random $r \in \mathbb{F}_q^\eta$,

$$\Pr\left[S(r) \cdot Q(r) = F(r) \cdot P(r) \mid SQ \neq FP\right] \leq \frac{m + w - 1}{q^\eta}$$

SDitH MPC protocol

- Principle: check $SQ = FP$ on t random points (SZ lemma)

1. Locally compute $[\![x_B]\!] = y - H' [\![x_A]\!]$
2. Locally compute $[\![S]\!]$ by Lagrange interpolation of $[\![x]\!] = ([\![x_A]\!] \mid [\![x_B]\!])$
3. Randomness oracle $\rightarrow$ $r_1, \ldots, r_t \in \mathbb{F}_q^\eta$
4. Locally compute $[\![S(r_i)]\!], [\![Q(r_i)]\!], F(r_i) \cdot [\![P(r_i)]\!] \quad \forall i \in [1:t]$
5. Check the product $S(r_i) \cdot Q(r_i) = F(r_i) \cdot P(r_i)$ from the shares

SDitH MPC protocol

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   - using [BN20] product-check protocol

- False positive probability: $p = \sum_{i=0}^{t} \binom{t}{i} \left(\frac{m+w-1}{q^\eta}\right)^i \left(1 - \frac{m+w-1}{q^\eta}\right)^{t-i} \left(\frac{1}{q^\eta}\right)^{t-i}$

SDitH signature scheme

Signature:

1. Generate random sharing $[\![x_A]\!]$, $[\![P]\!]$, $[\![Q]\!]$, $[\![a]\!]$, $[\![b]\!]$, $[\![c]\!]$
2. Commit the parties' shares:
$$[\![x_A]\!]_i, [\![P]\!]_i, [\![Q]\!]_i, [\![a]\!]_i, [\![b]\!]_i, [\![c]\!]_i \xrightarrow{\text{Commit}} \mathsf{com}_i$$
3. Derive the first challenge (randomness of MPC protocol):
$$\mathsf{com}_1, \dots, \mathsf{com}_N \xrightarrow{\text{Hash}} h_1 \to r, \varepsilon$$
4. Simulate the MPC protocol:
$$[\![x_A]\!], [\![P]\!], [\![Q]\!], [\![a]\!], [\![b]\!], [\![c]\!], r, \varepsilon \xrightarrow{\text{MPC}} [\![\alpha]\!], [\![\beta]\!], [\![v]\!]$$
5. Derive the second challenge (index of non-opened party):
$$h_1, [\![\alpha]\!], [\![\beta]\!], [\![v]\!] \xrightarrow{\text{Hash}} h_2 \to I$$
6. Build the signature from
$$h_1,\ h_2,\ \left\{[\![x_A]\!]_i, [\![P]\!]_i, [\![Q]\!]_i, [\![a]\!]_i, [\![b]\!]_i, [\![c]\!]_i\right\}_{i \in I},\ \{\mathsf{com}_i, [\![\alpha]\!]_i, [\![\beta]\!]_i, [\![v]\!]_i\}_{i \notin I}$$

SDitH signature scheme

Signature:

1. Generate random sharing $[\![x_A]\!], [\![P]\!], [\![Q]\!], [\![a]\!], [\![b]\!], [\![c]\!]$
2. Commit the parties' shares:

$$[\![x_A]\!]_i, [\![P]\!]_i, [\![Q]\!]_i, [\![a]\!]_i, [\![b]\!]_i, [\![c]\!]_i \xrightarrow{\text{Commit}} \mathsf{com}_i$$

$\times\,\tau$

3. Derive the first challenge (randomness of MPC protocol):

$$\mathsf{com}_1, \ldots, \mathsf{com}_N \xrightarrow{\text{Hash}} h_1 \to r, \varepsilon$$

$\times\,\tau$

4. Simulate the MPC protocol:

$$[\![x_A]\!], [\![P]\!], [\![Q]\!], [\![a]\!], [\![b]\!], [\![c]\!], r, \varepsilon \xrightarrow{\text{MPC}} [\![\alpha]\!], [\![\beta]\!], [\![v]\!]$$

$\times\,\tau$

5. Derive the second challenge (index of non-opened party):

$$h_1, [\![\alpha]\!], [\![\beta]\!], [\![v]\!] \xrightarrow{\text{Hash}} h_2 \to I$$

$\times\,\tau$

6. Build the signature from

$\times\,\tau$

$$h_1,\ h_2,\ \{[\![x_A]\!]_i, [\![P]\!]_i, [\![Q]\!]_i, [\![a]\!]_i, [\![b]\!]_i, [\![c]\!]_i\}_{i \in I},\ \{\mathsf{com}_i, [\![\alpha]\!]_i, [\![\beta]\!]_i, [\![v]\!]_i\}_{i \notin I}$$

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hypercube $\rightarrow$ $\log N + 1$ party emulations

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SDitH signature scheme

Parameter Set	MPCitH Parameters						Sizes (in bytes)			
	N	ℓ	τ	η	t	p	pk	sk	Sig. Avg	Sig. Max
SDitH-L1-hyp	2^8	–	17	4	3	$2^{-70.6}$	132	432	8 476	8 496
SDitH-L3-hyp	2^8	–	26	4	3	$2^{-71.8}$	180	628	19 498	19 544
SDitH-L5-hyp	2^8	–	34	4	4	$2^{-94.2}$	244	838	33 843	33 924

Instance	KeyGen		Sign		Verify	
	ms	cycles	sign ms	cycles	verify ms	cycles
SDitH-gf256-L1-hyp	5.47	14.2M	4.18	10.8M	3.74	9.7M
SDitH-gf256-L3-hyp	6.41	16.6M	10.13	26.2M	8.83	22.9M
SDitH-gf256-L5-hyp	11.06	28.7M	19.25	49.9M	16.98	44.0M
SDitH-gf251-L1-hyp	3.05	7.9M	8.17	21.2M	7.83	20.3M
SDitH-gf251-L3-hyp	3.67	9.5M	17.98	46.6M	17.08	44.3M
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SDitH signature scheme

☝️ ∃ variant based in MPCitH with threshold secret sharing

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MPC in the Head with Threshold Secret Sharing (a.k.a. TCitH)

Background: Shamir's secret sharing

$$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$$

- Generate
  - Let $(r_1, \ldots, r_\ell) \leftarrow \$$
  - Let P the polynomial of coefficients $(x, r_1, \ldots, r_\ell)$

$$\begin{cases} [\![x]\!]_1 = P(f_1) \\ \quad\vdots \\ [\![x]\!]_N = P(f_N) \end{cases} \qquad \text{with} \quad f_1, \ldots, f_N \in \mathbb{F} \text{ distinct field elements}$$

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- Reconstruct
  - Interpolate P from $[\![x]\!]_1, \ldots, [\![x]\!]_N$
  - $x = P(0)$

Background: Shamir's secret sharing

- $(\ell + 1, N)$-**threshold** linear secret sharing scheme (LSSS)
  - Linearity: $[\![x]\!] + [\![y]\!] = [\![x + y]\!]$

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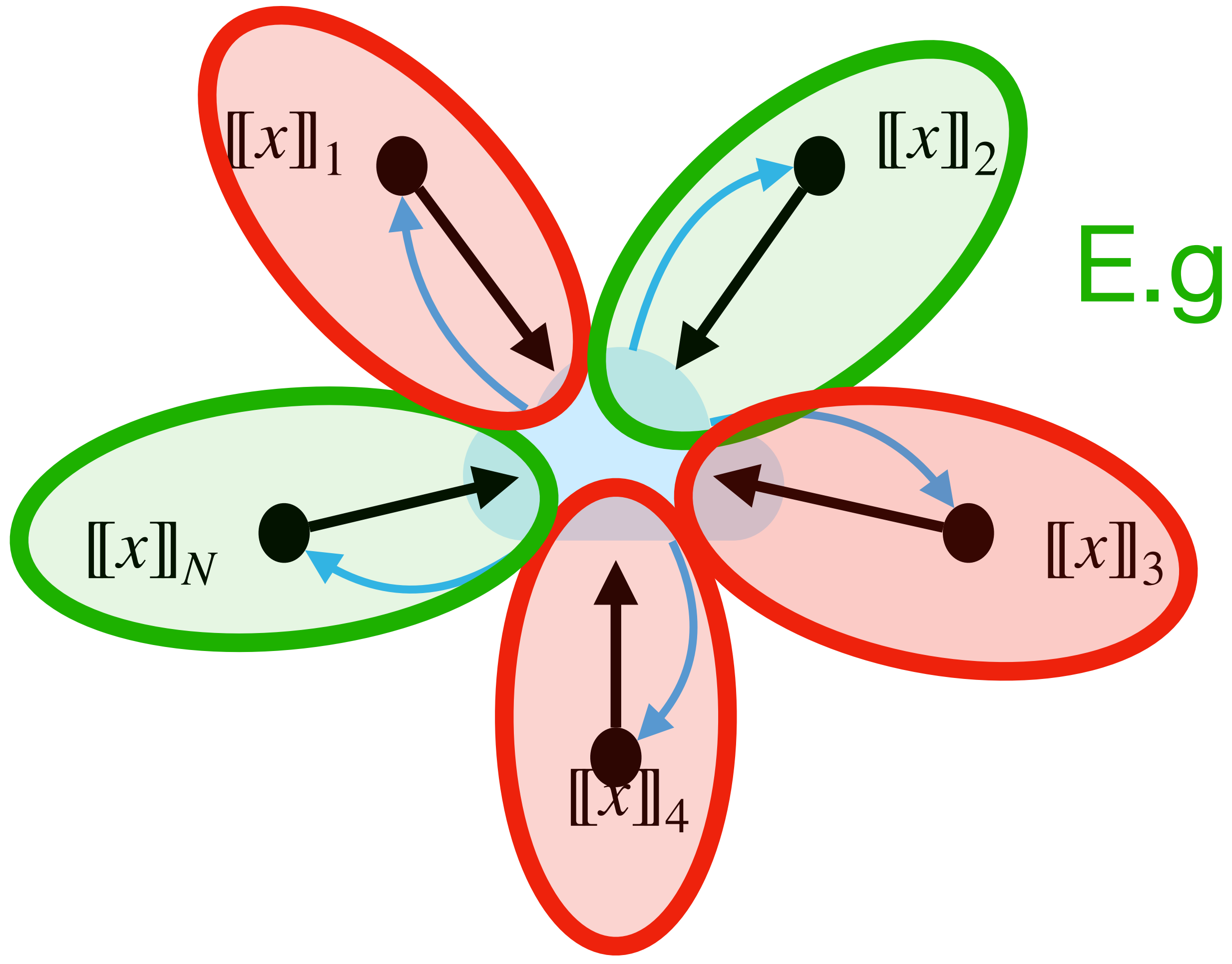
  - Any set of ℓ shares is random and independent of x
  - Any set of $\ell + 1$ shares → coefficients $(x, r_1, \ldots, r_\ell)$ → all the shares
- $[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$ is a **Reed-Solomon codeword** of $(x, r_1, \ldots, r_\ell)$

MPCitH with threshold LSSS (a.k.a TCitH)

- **[FR23]** Feneuil, Rivain. "Threshold Linear Secret Sharing to the Rescue of MPC-in-the-Head" (Asiacrypt 2023)
- ZK property $\Rightarrow$ only open ℓ parties
  - Verifier challenges a set $I \subseteq \{1,\ldots,N\}$ s.t. $|I| = \ell$
  - Prover opens $\{[\![x]\!]_i, \rho_i\}_{i \in I}$



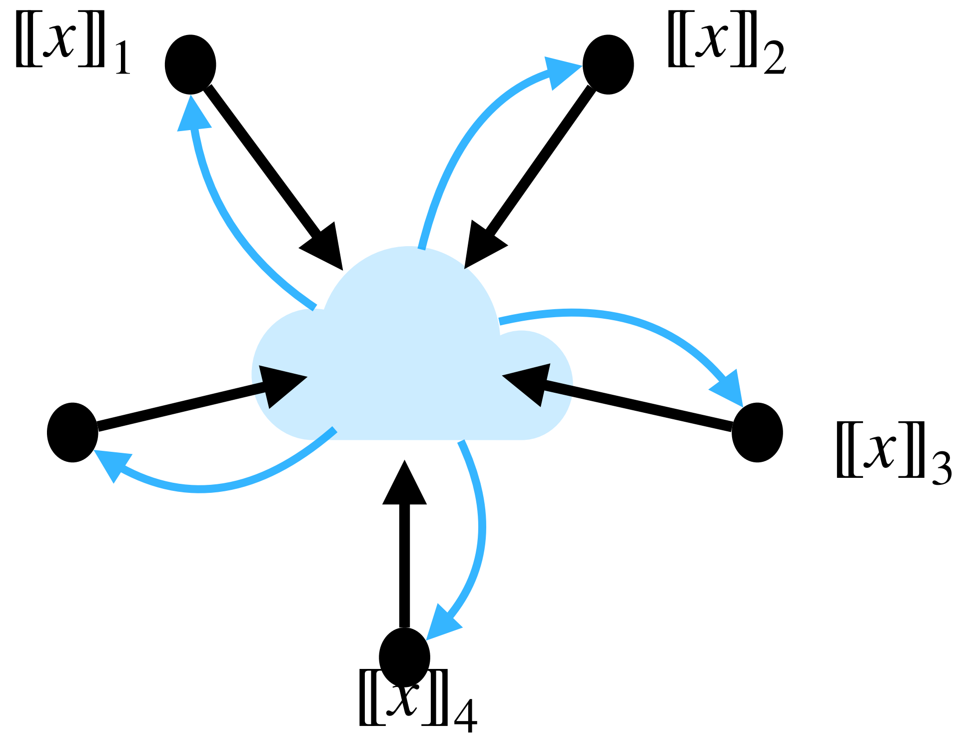
E.g. $\ell = 2$

MPCitH with threshold LSSS (a.k.a TCitH)

① Generate and commit shares

$[\![x]\!] = ([\![x]\!]_1, \ldots, [\![x]\!]_N)$

② Run MPC in their head



④ Open parties in I

$\mathrm{Com}^{\rho_1}([\![x]\!]_1)$

$\cdots$

$\mathrm{Com}^{\rho_N}([\![x]\!]_N)$

send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$

I

$([\![x]\!]_i, \rho_i)_{i \in I}$

③ Chose random set of parties

$I \subseteq \{1,\ldots,N\}$, s.t. $|I| = \ell$

⑤ Check $\forall i \in I$

- Commitments $\mathrm{Com}^{\rho_i}([\![x]\!]_i)$
- MPC computation $[\![\alpha]\!]_i = \varphi([\![x]\!]_i)$

Check $g(y, \alpha) =$ Accept

Prover

Verifier

MPCitH with threshold LSSS (a.k.a TCitH)

Threshold LSSS $\Rightarrow$ *cannot generate shares from seeds*

Prover

① Generate and commit shares
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$\cdots$
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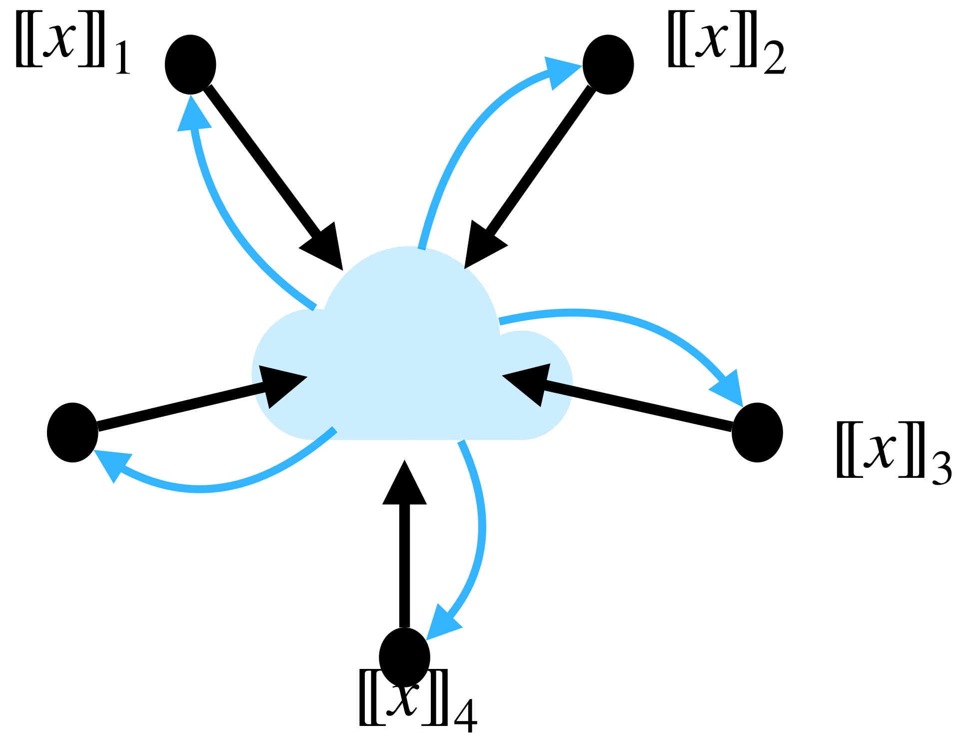

send broadcast
$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$ →

Verifier

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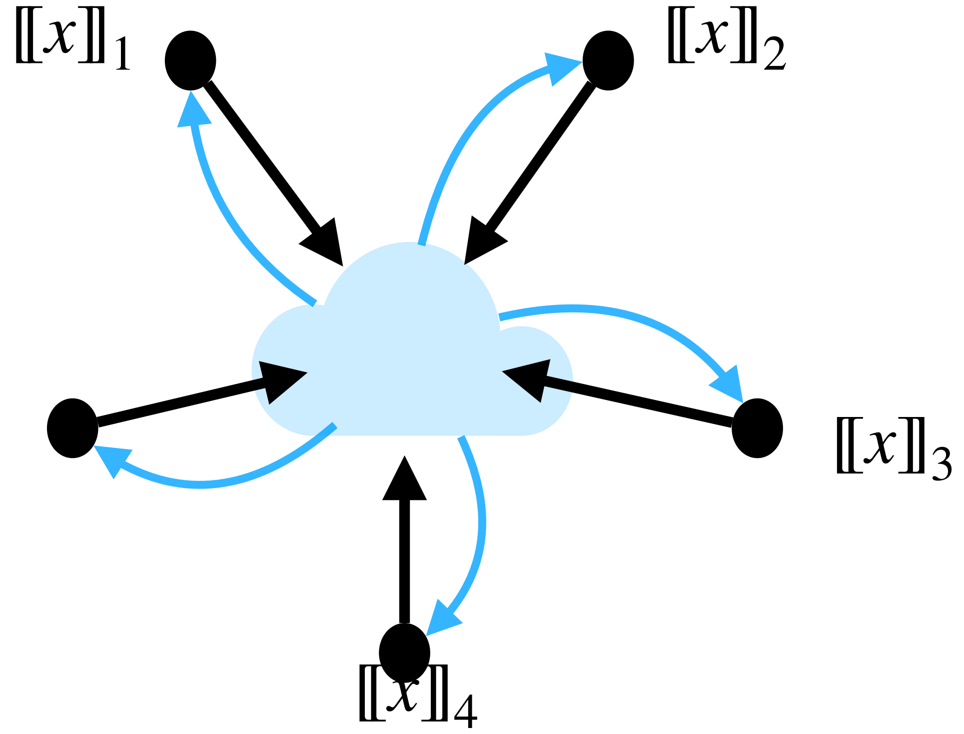

send broadcast

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$[\![\alpha]\!]$ is an RS codeword $\Rightarrow \ell + 1$ shares fully determine the sharing

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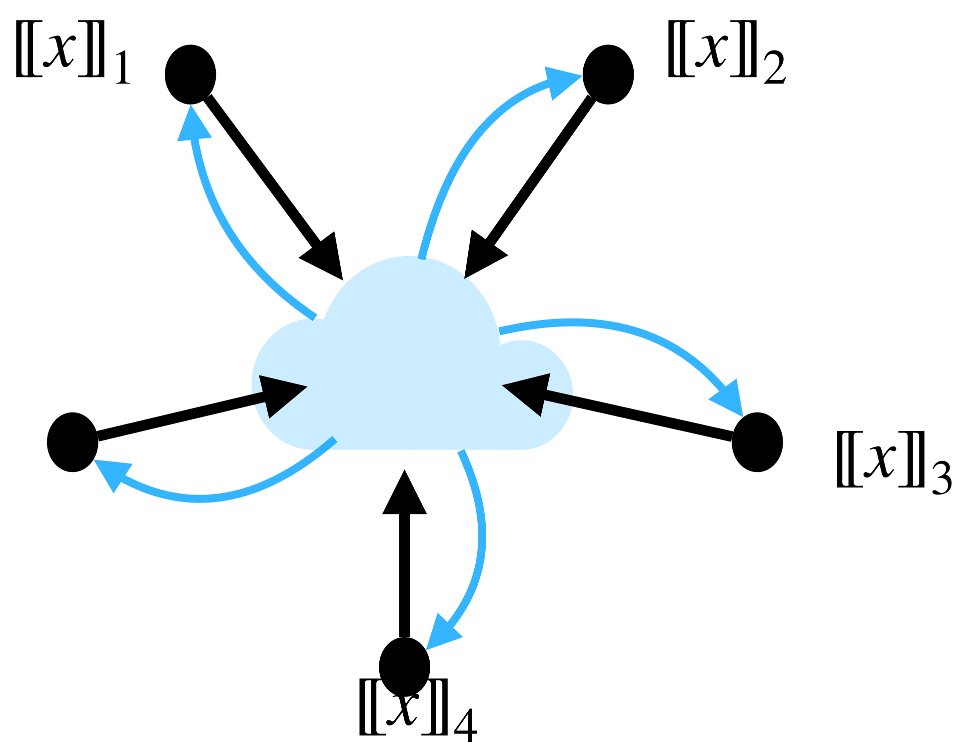

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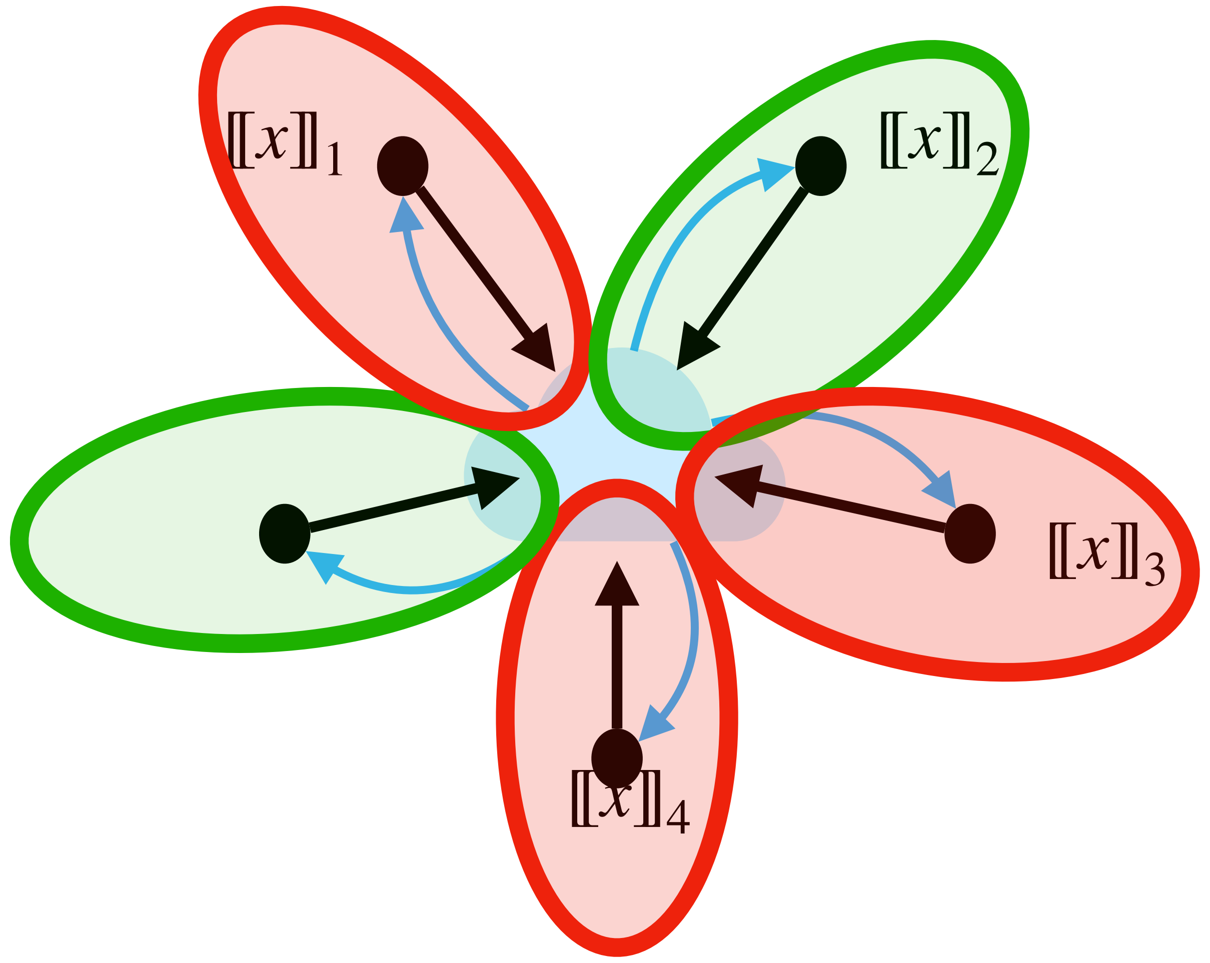

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ℓ parties opened instead of $N - 1$

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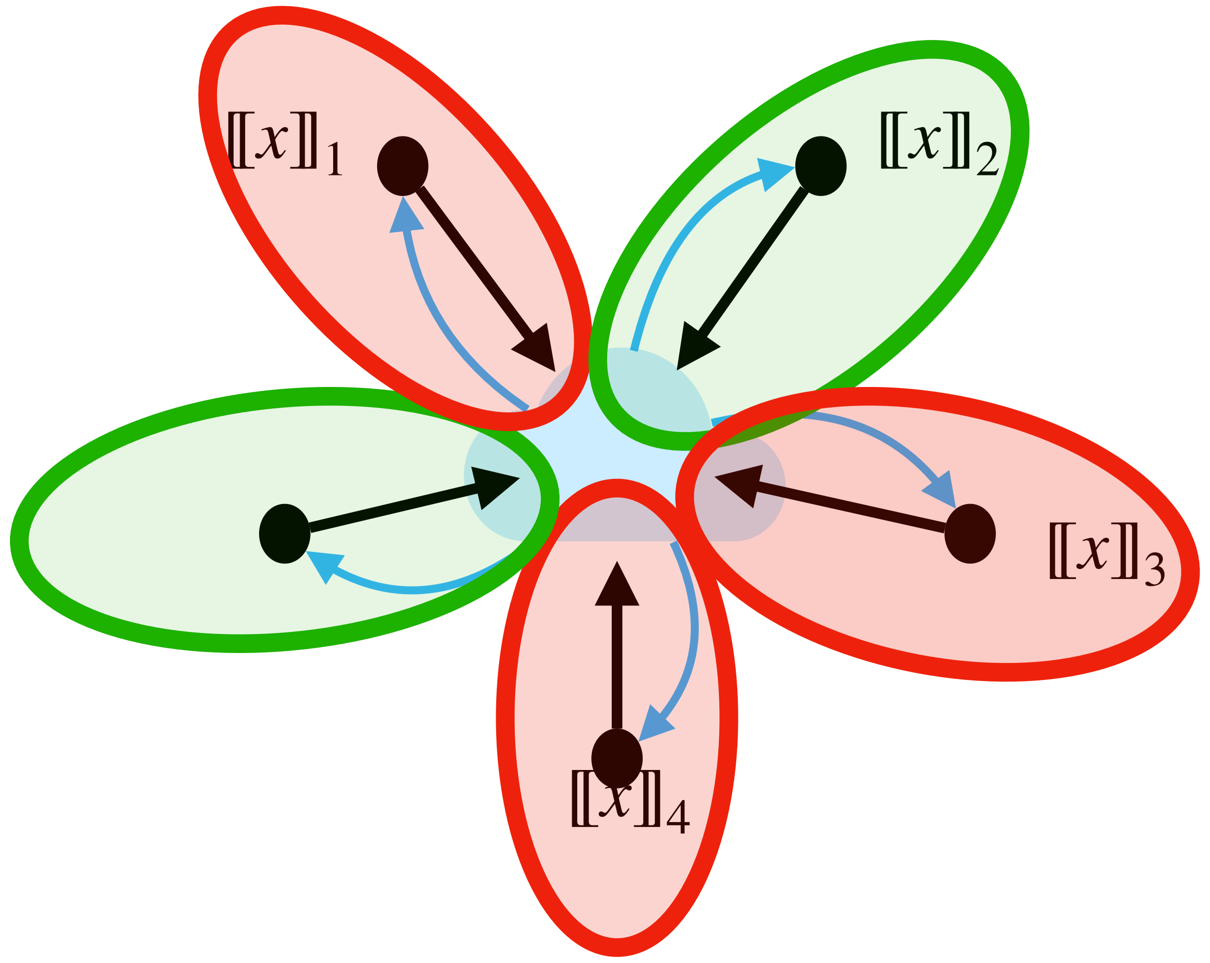

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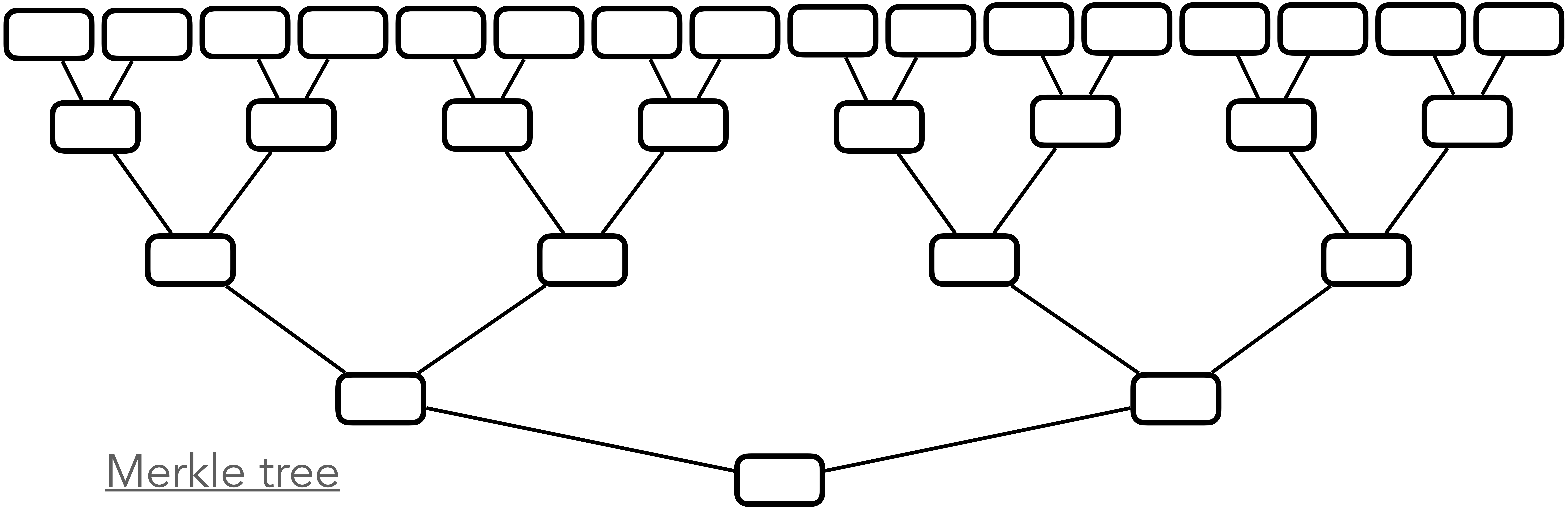
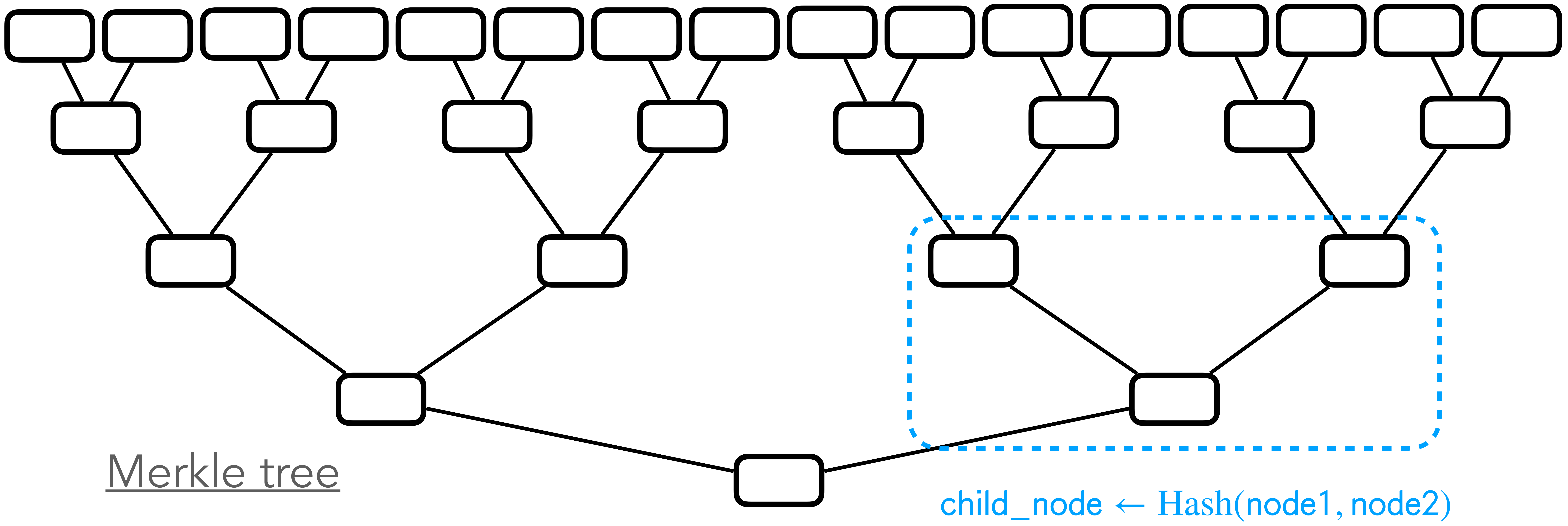
Sharing and commitments



Merkle tree

Sharing and commitments



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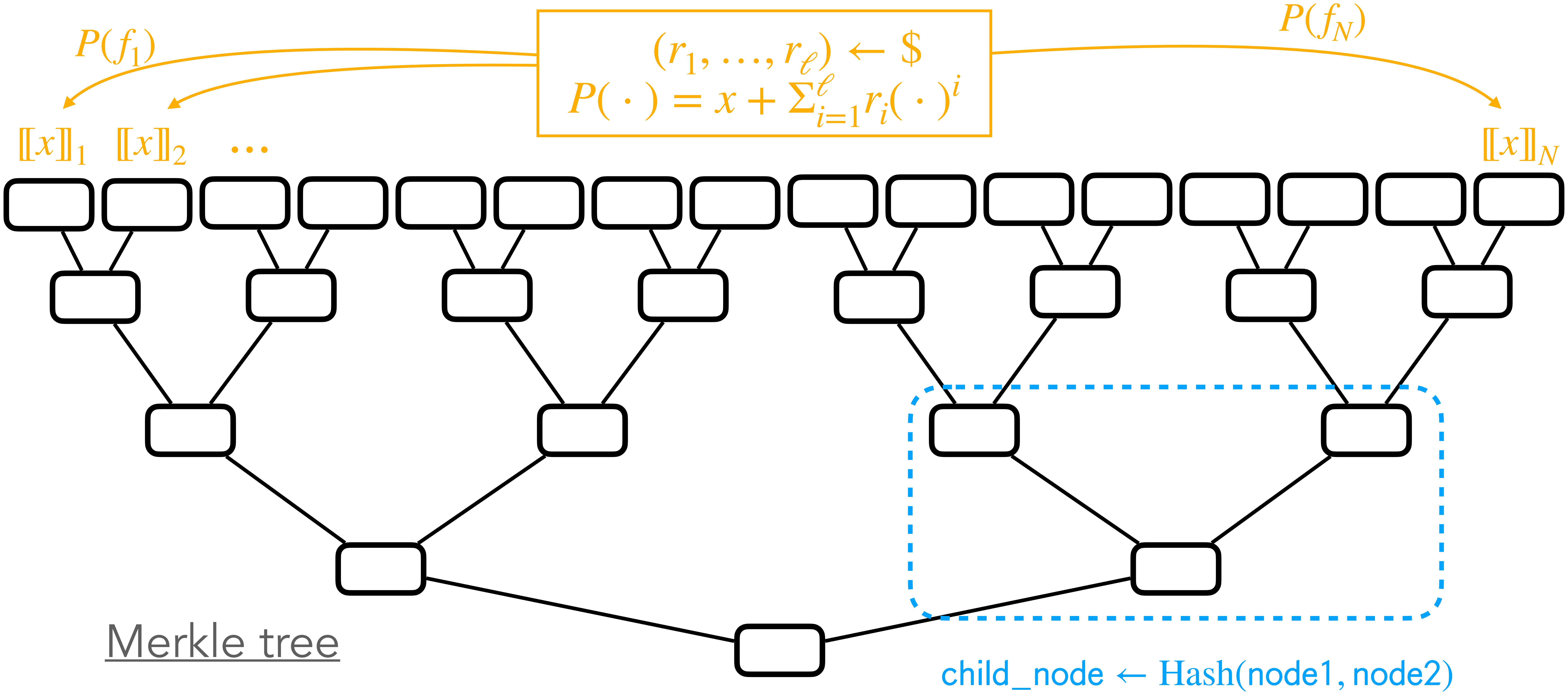

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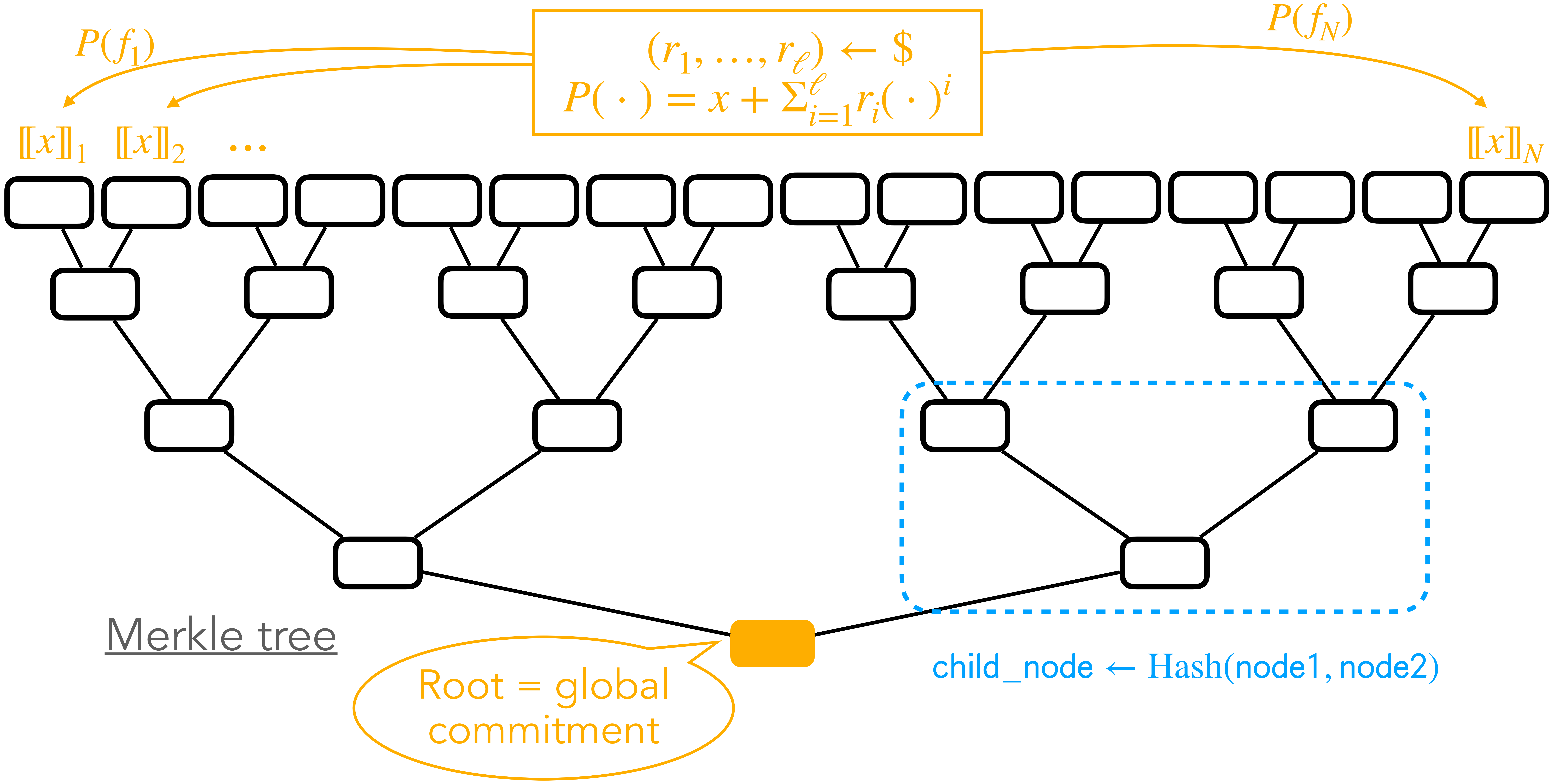

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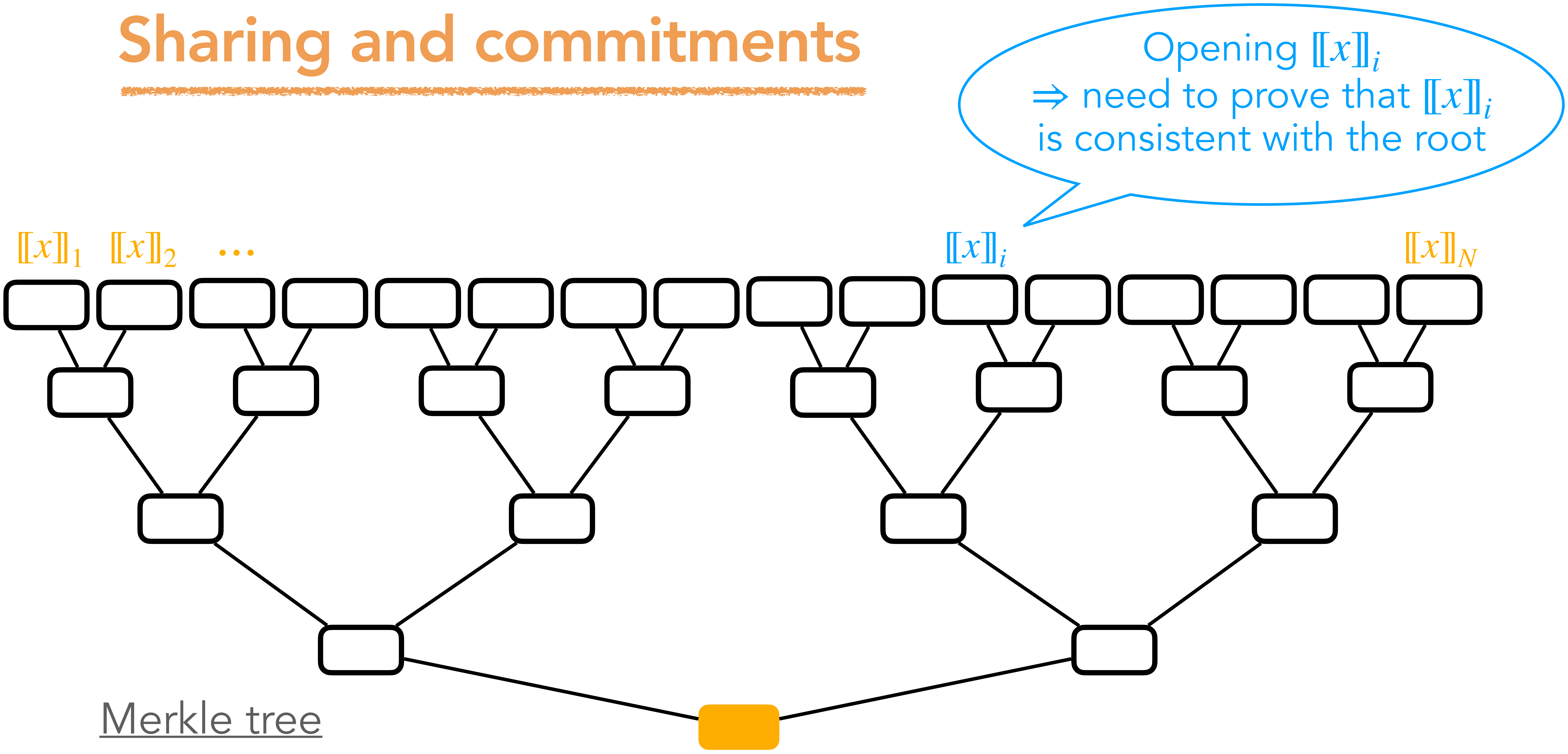

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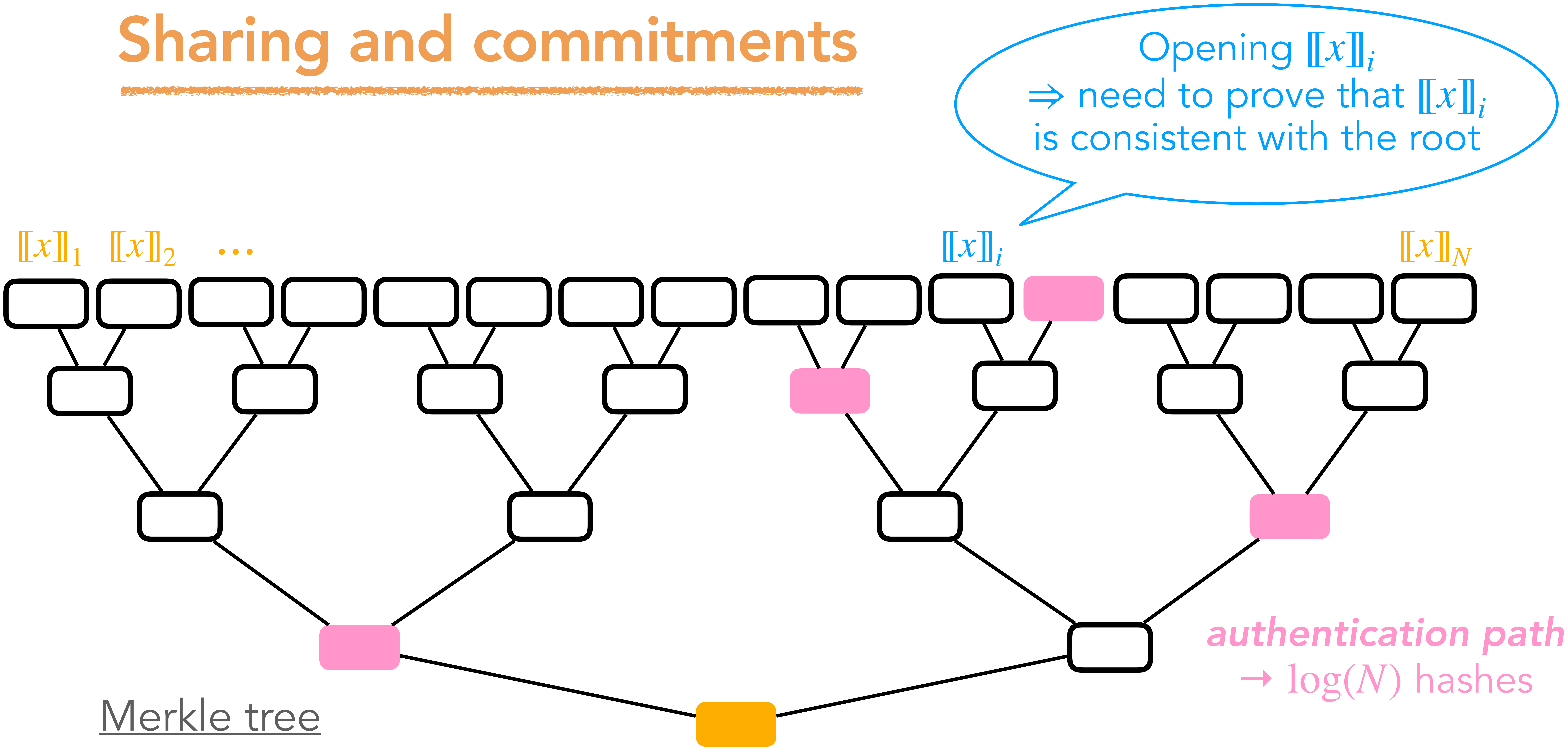

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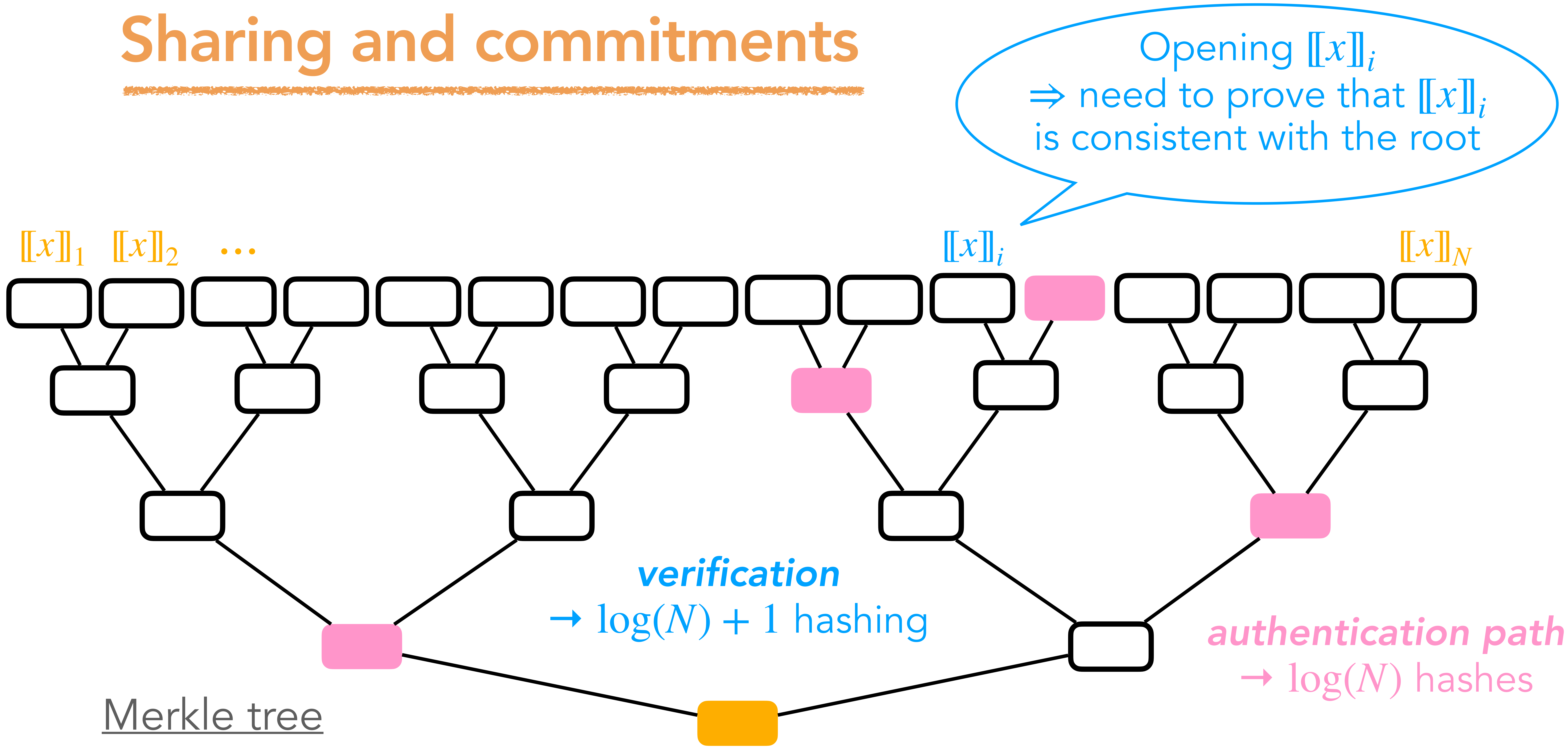

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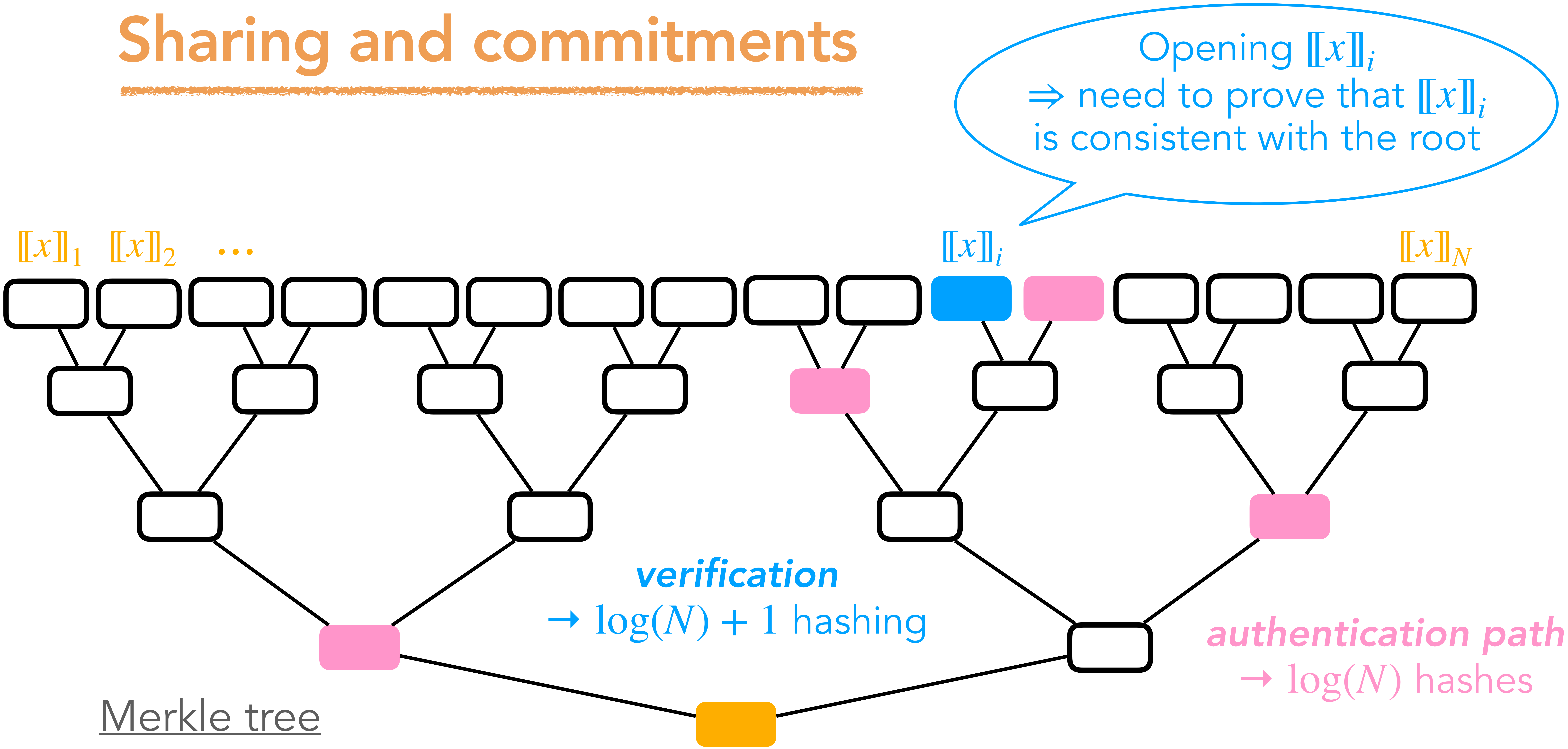
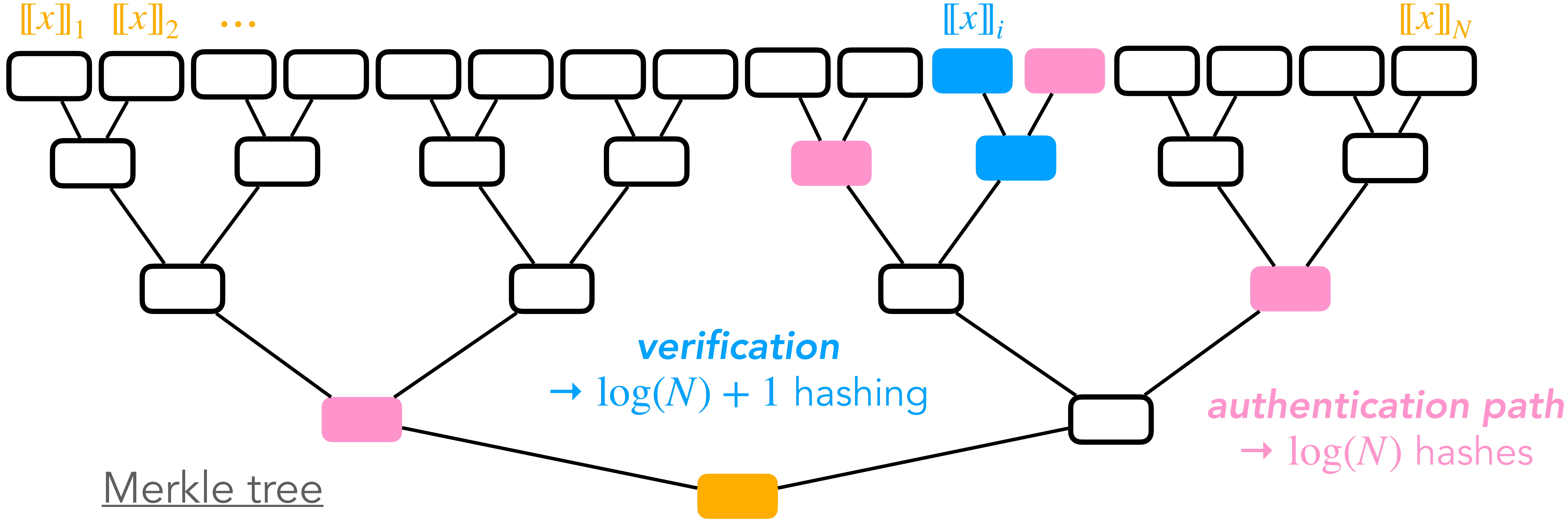


Sharing and commitments

Opening $[\![x]\!]_i$
$\Rightarrow$ need to prove that $[\![x]\!]_i$ is consistent with the root



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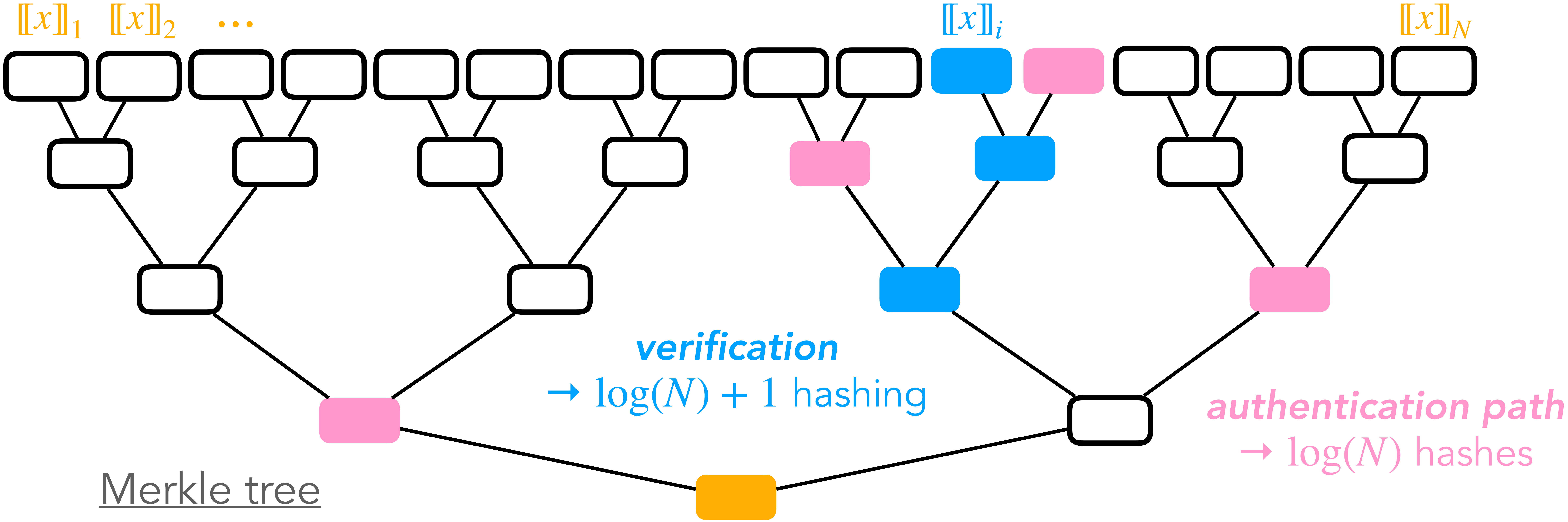

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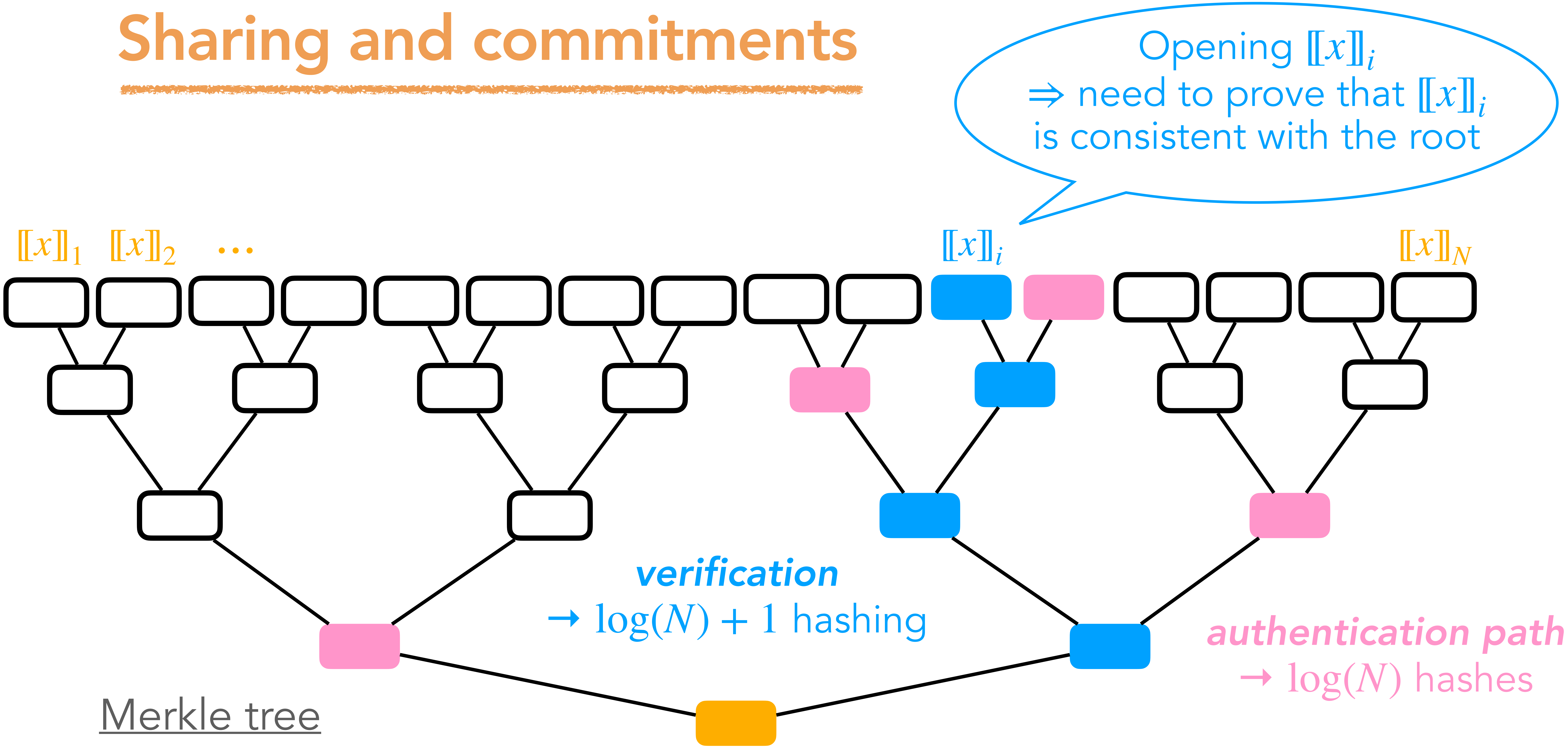

MPCitH with threshold LSSS (a.k.a TCitH)

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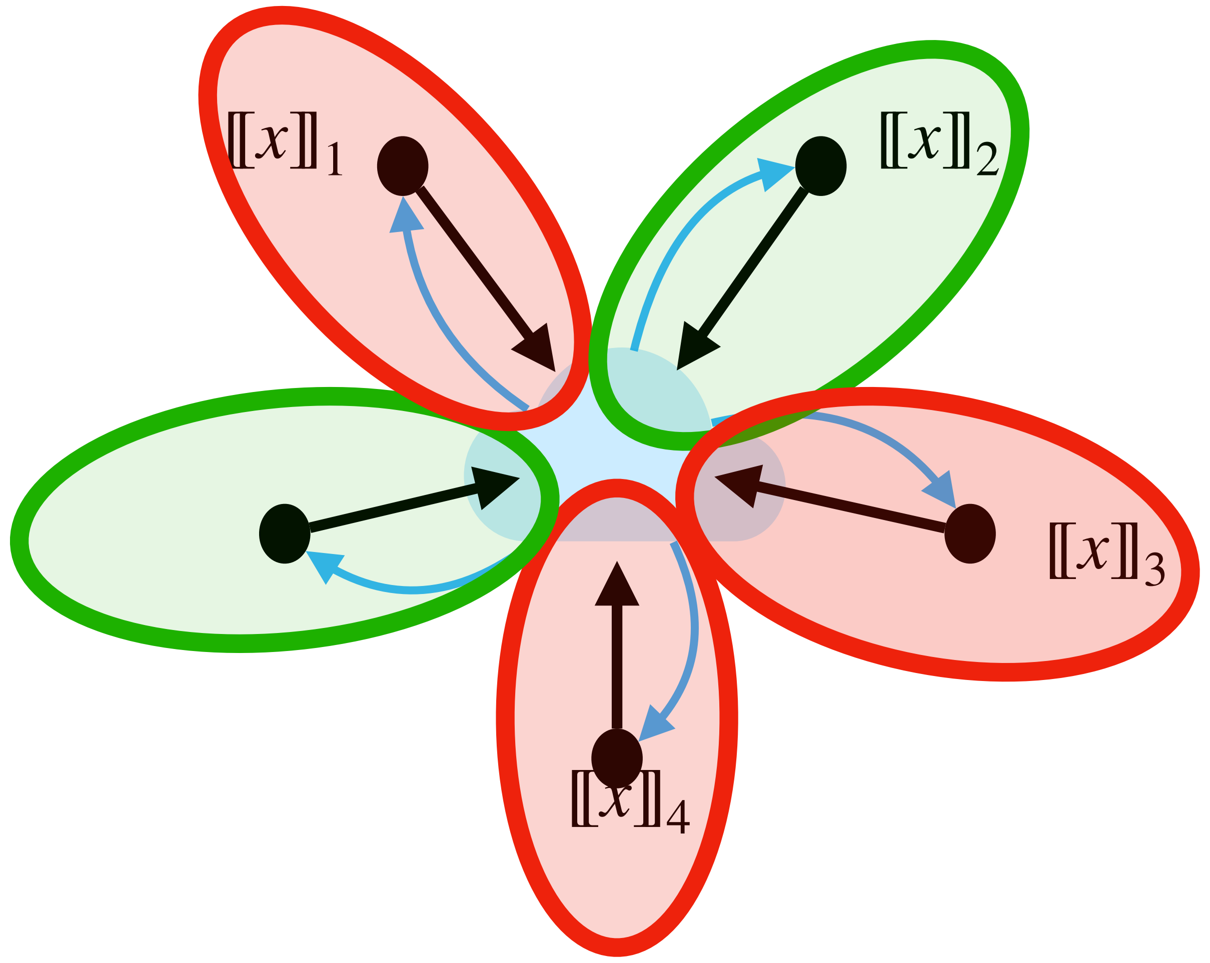

④ Open parties in I

Merkle root →

send broadcast

$[\![\alpha]\!]_1, \ldots, [\![\alpha]\!]_N$ →

← I

$\{[\![x]\!]_i \boxed{\text{auth}_i}\}_{i \in I}$ →

authentication path

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⑤ Check $\forall i \in I$

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- One would expect:

$$P[\text{cheat detected}] = \frac{\ell}{N} \quad \Rightarrow \quad \boxed{\text{Soundness error} = 1 - \frac{\ell}{N}}$$

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🤔 *not really good*

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- But the verifier also check broadcast sharings $[\![\alpha]\!]$
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  - i.e. valid Reed-Solomon codewords

    $\Rightarrow$ limits the cheating possibilities

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with false positives

$$\boxed{\frac{1}{\binom{N}{\ell}} + p \cdot \frac{\ell(N-\ell)}{\ell+1}}$$

Soundness

- One would expect:

$$P[\text{cheat detected}] = \frac{\ell}{N} \quad \Rightarrow \quad \boxed{\cancel{\text{Soundness error} = 1 - \frac{\ell}{N}}}$$

- But the verifier also check broadcast sharings $[\![\alpha]\!]$
  - must be valid Shamir's secret sharings
  - i.e. valid Reed-Solomon codewords

  $\Rightarrow$ limits the cheating possibilities

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not so great

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$$\Longrightarrow \boxed{\frac{1}{\binom{N}{\ell}} + p}$$

🤩

Comparison

$\ell = 1 \;\Rightarrow\;$ Similar soundness: $\dfrac{1}{N} + p$ 🤝

	MPCitH + seed trees + hypercube [AGHHJY23]	**TCitH** $\ell = 1$
Prover runtime	Party emulations: $\log N + 1$ Symmetric crypto: $O(N)$	Party emulations: 2 Symmetric crypto: $O(N)$

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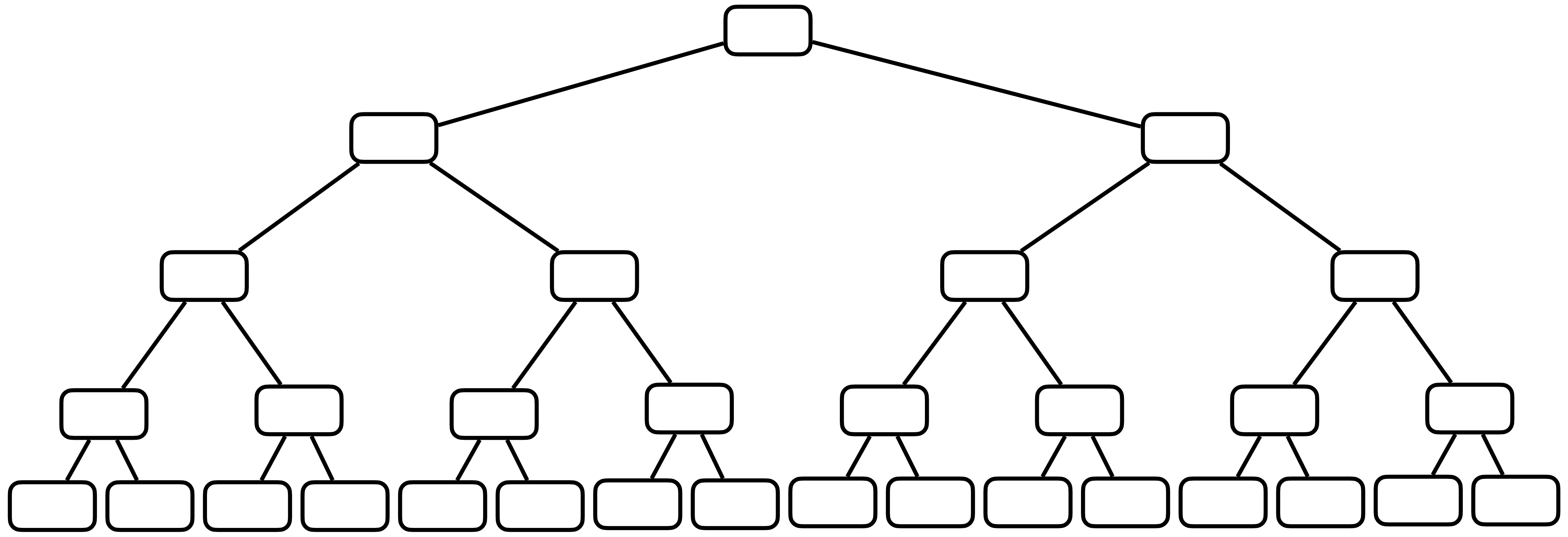
Getting rid of these limitations

→ TCitH with GGM tree

TCitH with GGM trees

<u>Step 1</u>: Generate a replicated secret sharing of x [ISN89]



$$r_1 + r_2 + \quad \cdots \qquad\qquad \cdots \; + r_N \;=\; x + \Delta_x$$

TCitH with GGM trees

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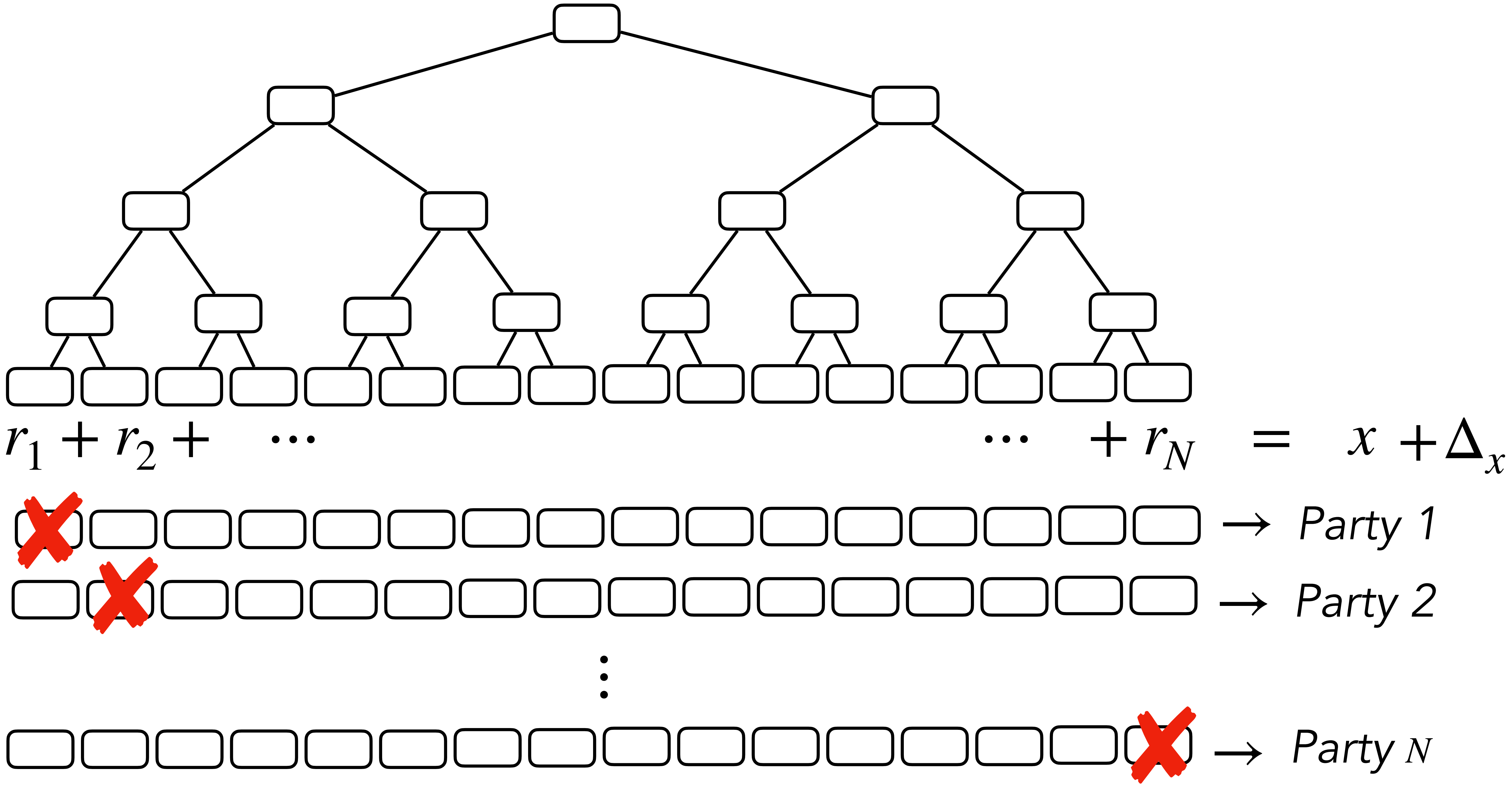

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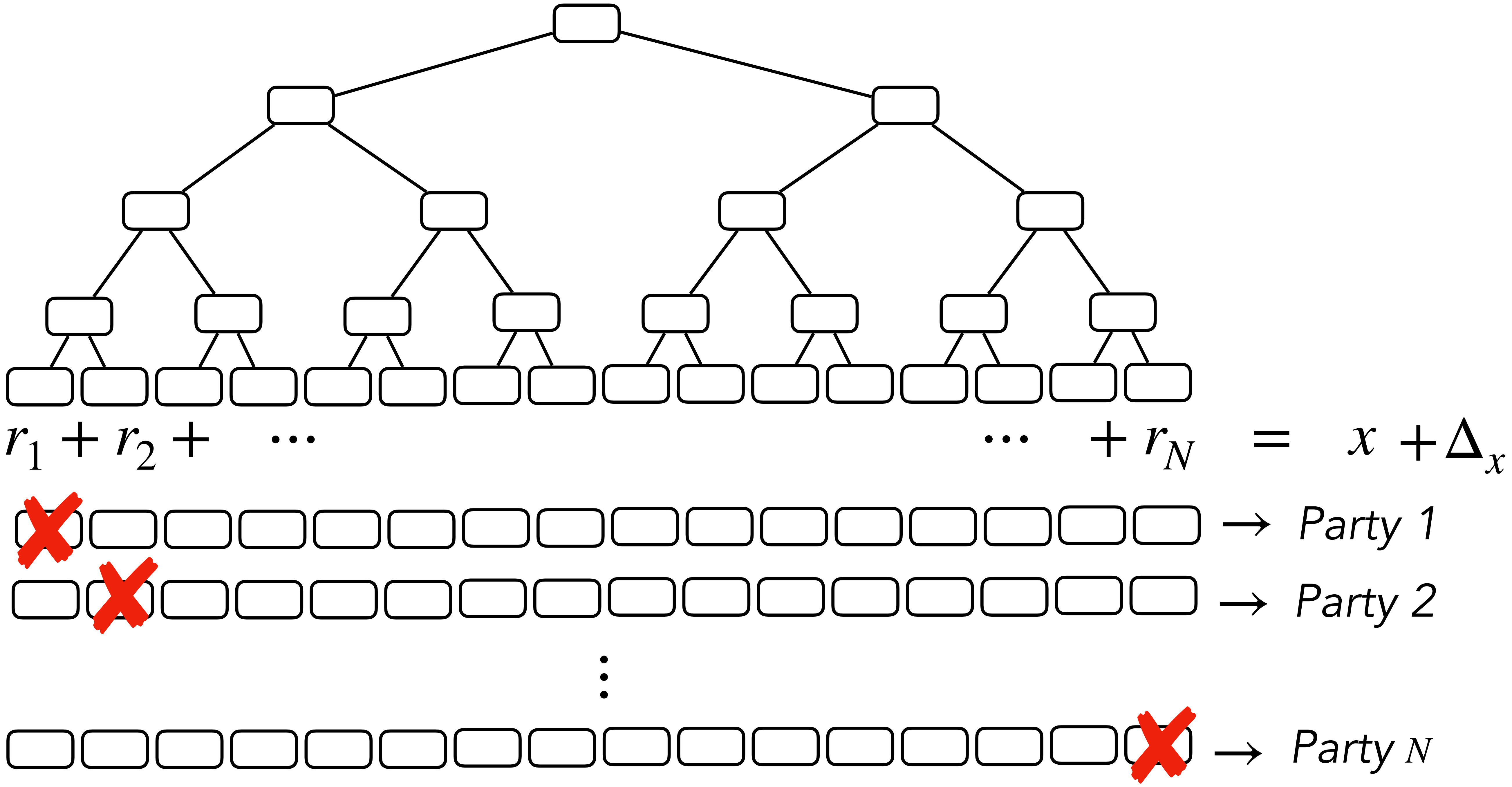

Step 2: Convert it into a Shamir's secret sharing [CDI05]

Let $P(X) = \Delta_x + \sum_j r_j P_j(X)$

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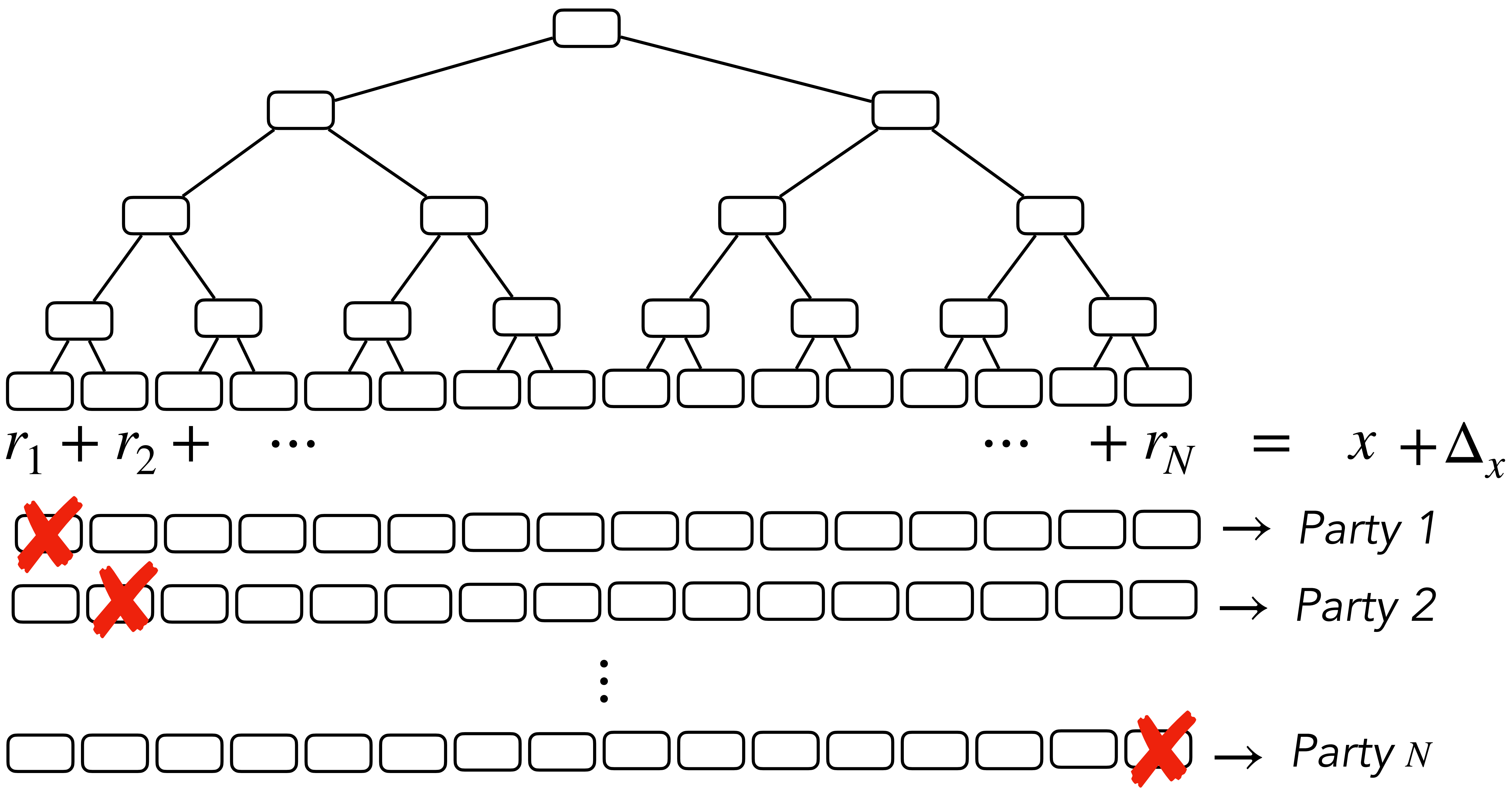

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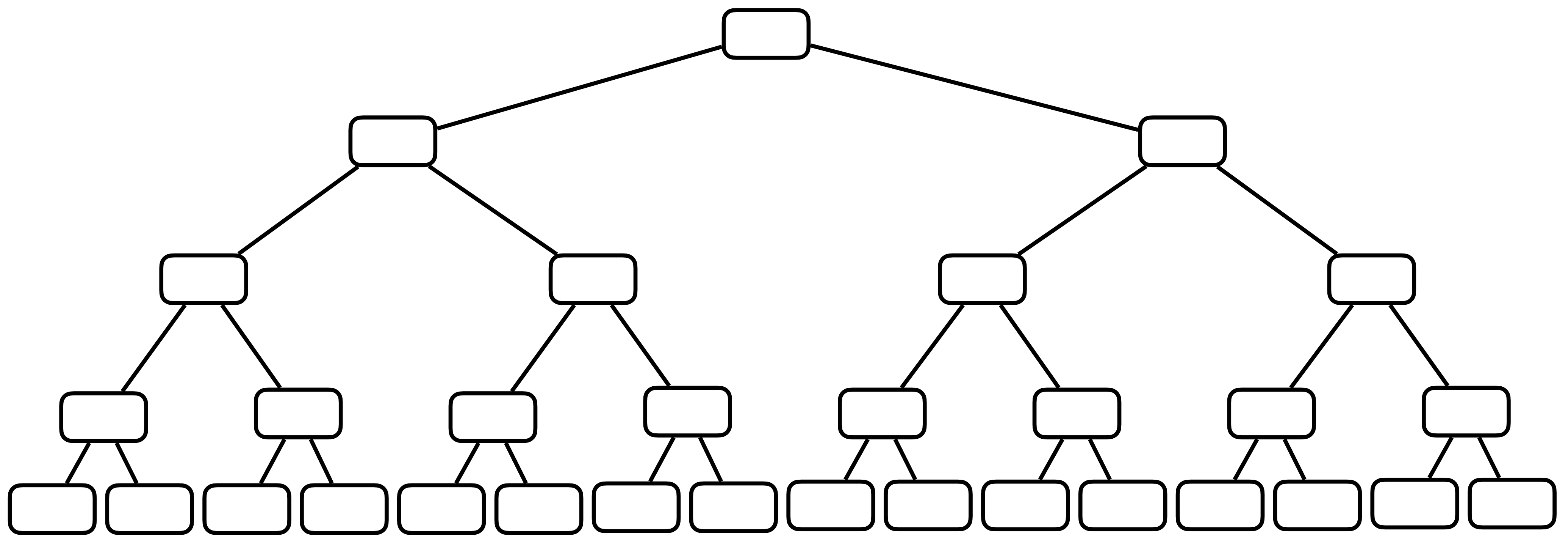
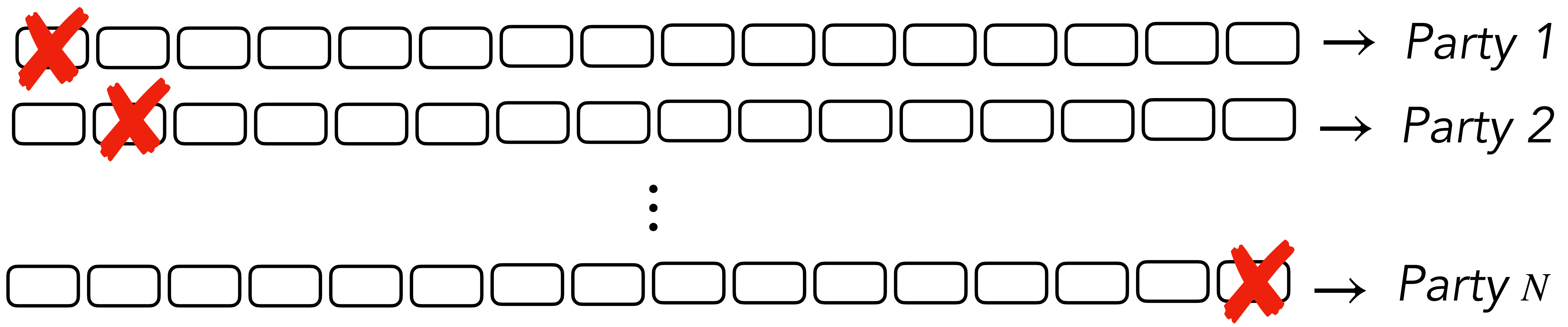


$$r_1 + r_2 + \cdots \cdots + r_N = x + \Delta_x$$



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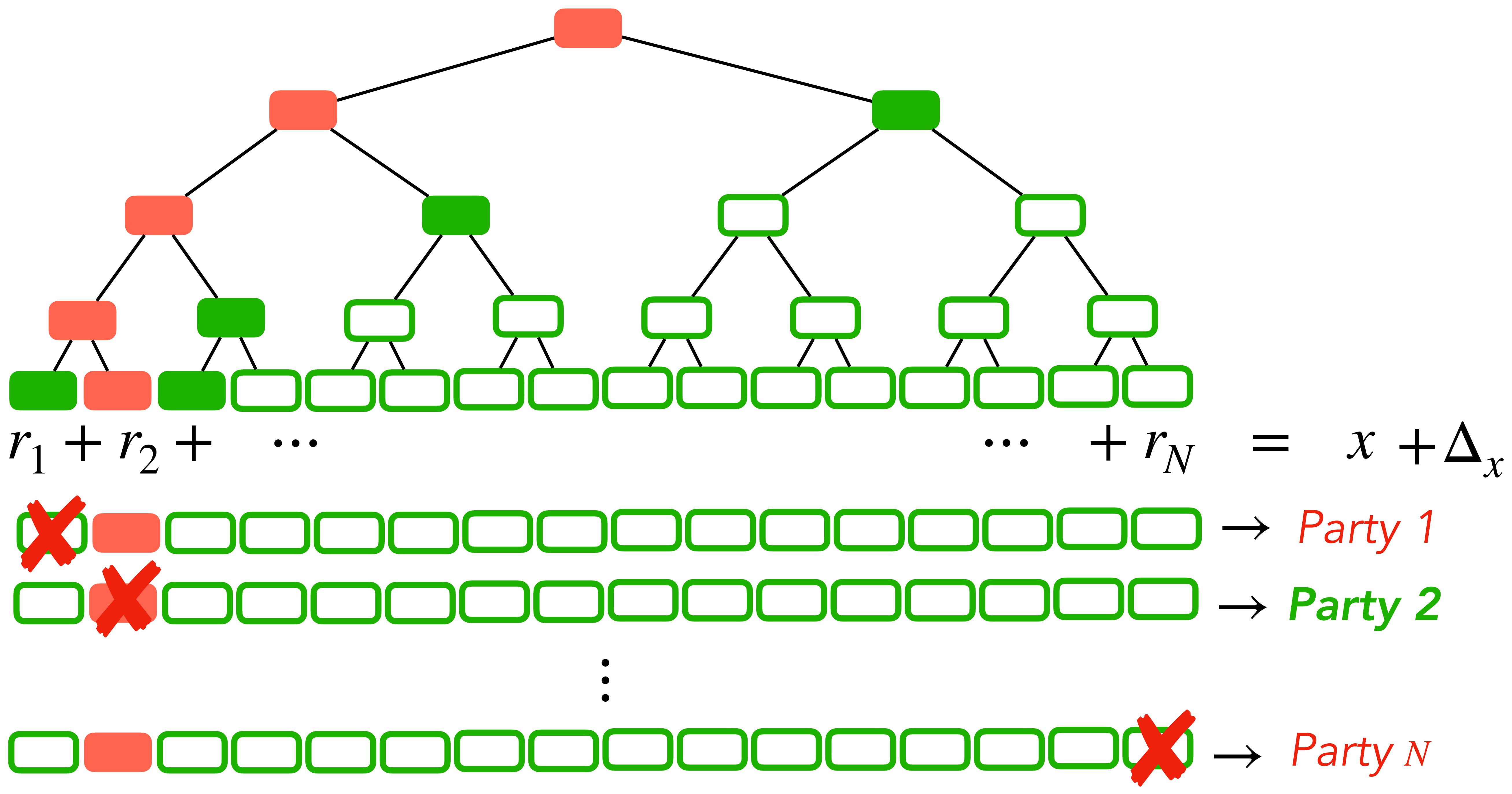

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🐌 Loose fast verification

Using multiplication homomorphism

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$$[\![x]\!]^{(d)} \cdot [\![y]\!]^{(d)} = [\![x \cdot y]\!]^{(2d)}$$

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false positive proba $1/|\mathbb{F}|$

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Soundness error

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Here: $\ell \cdot \deg f_j$ $\left(\frac{1}{|\mathbb{F}|}\right)^{\#\alpha}$

$$\frac{\binom{d_\alpha}{\ell}}{\binom{N}{\ell}} + p$$

Soundness error

Signature from MQ and TCitH

MQ Problem

- Parameters
  - A field $\mathbb{F}_q$, $n \in \mathbb{N}$ (# variables) , $m \in \mathbb{N}$ (# equations)
- Let
  - $x \leftarrow \mathbb{F}_q^n$ *(MQ solution)*
  - $A_i \leftarrow \mathbb{F}_q^{n \times n} \quad \forall i \in [1:m]$ *(m random matrices)*
  - $b_i \leftarrow \mathbb{F}_q^n \quad \forall i \in [1:m]$ *(m random vectors)*
  - $y = (y_1, \ldots, y_m) \in \mathbb{F}_q^m$ s.t. $\begin{cases} y_1 &= x^T A_1 x + b_1^T x \\ &\vdots \\ y_m &= x^T A_m x + b_m^T x \end{cases}$
- From $\big(\{A_i\}, \{b_i\}, y\big)$ find x

Checking a MQ instance = checking m quadratic constraints on the secret x

We can directly apply the previous protocol

$\Downarrow$

|sig| $\approx$ 3 kB

Shorter Signatures from TCitH-GGM

	Original Size	*Our Variant*	*Saving*
Biscuit	4 758 B	4 048 B	-15 %
MIRA	5 640 B	5 340 B	-5 %
MiRitH-Ia	5 665 B	4 694 B	-17 %
MiRitH-Ib	6 298 B	5 245 B	-17 %
MQOM-31	6 328 B	4 027 B	-37 %
MQOM-251	6 575 B	4 257 B	-35 %
RYDE	5 956 B	5 281 B	-11 %
SDitH	8 241 B	7 335 B	-27 %
MQ over GF(4)	8 609 B	3 858 B	-55 %
SD over GF(2)	11 160 B	7 354 B	-34 %
SD over GF(2)	12 066 B	6 974 B	-42 %

* $N = 256$

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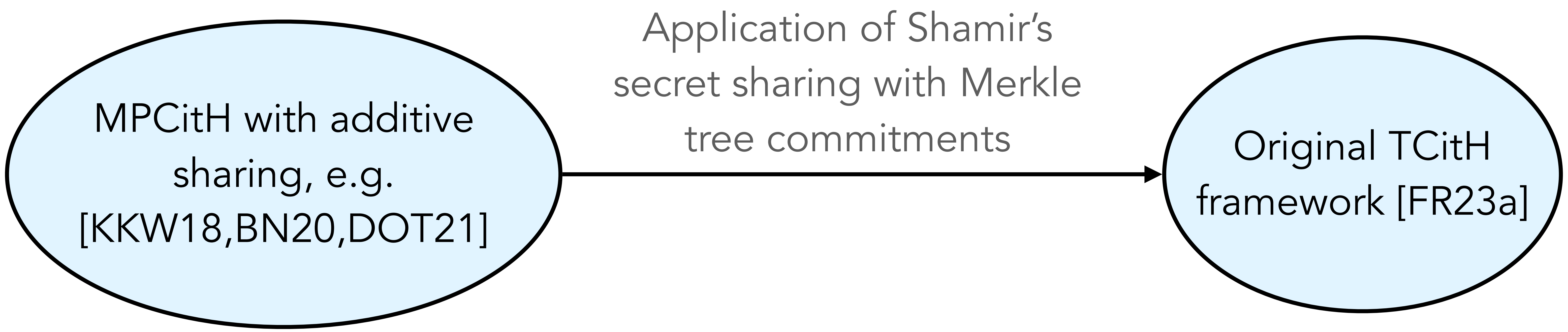
* $N = 256$ * $N = 2048$

Shorter Signatures from TCitH-GGM

Two very recent works :

- [BBMO+24] Baum, Beullens, Mukherjee, Orsini, Ramacher, Rechberger, Roy, Scholl. *One Tree to Rule Them All: Optimizing GGM Trees and OWFs for Post-Quantum Signatures.* https://ia.cr/2024/490
  - General techniques to reduce the size of GGM trees: **tree merging** & **proof of work**
  - **Apply to TCitH-GGM** (gain of ~500 B at 128-bit security)
- [BFGNR24] Bidoux, Feneuil, Gaborit, Neveu, Rivain. *Dual Support Decomposition in the Head: Shorter Signatures from Rank SD and MinRank.* https://ia.cr/2024/541
  - New MPC protocols for TCitH / VOLEitH signatures based on **MinRank & Rank SD**

Connection to other proof systems



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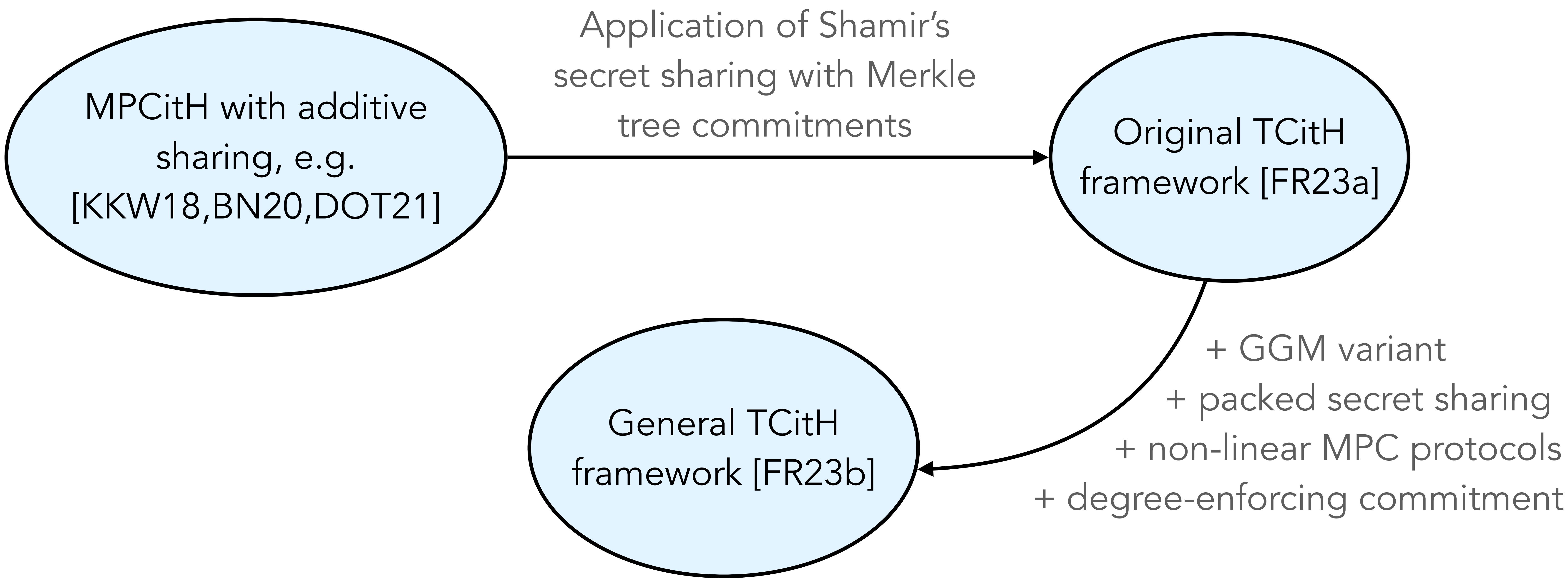

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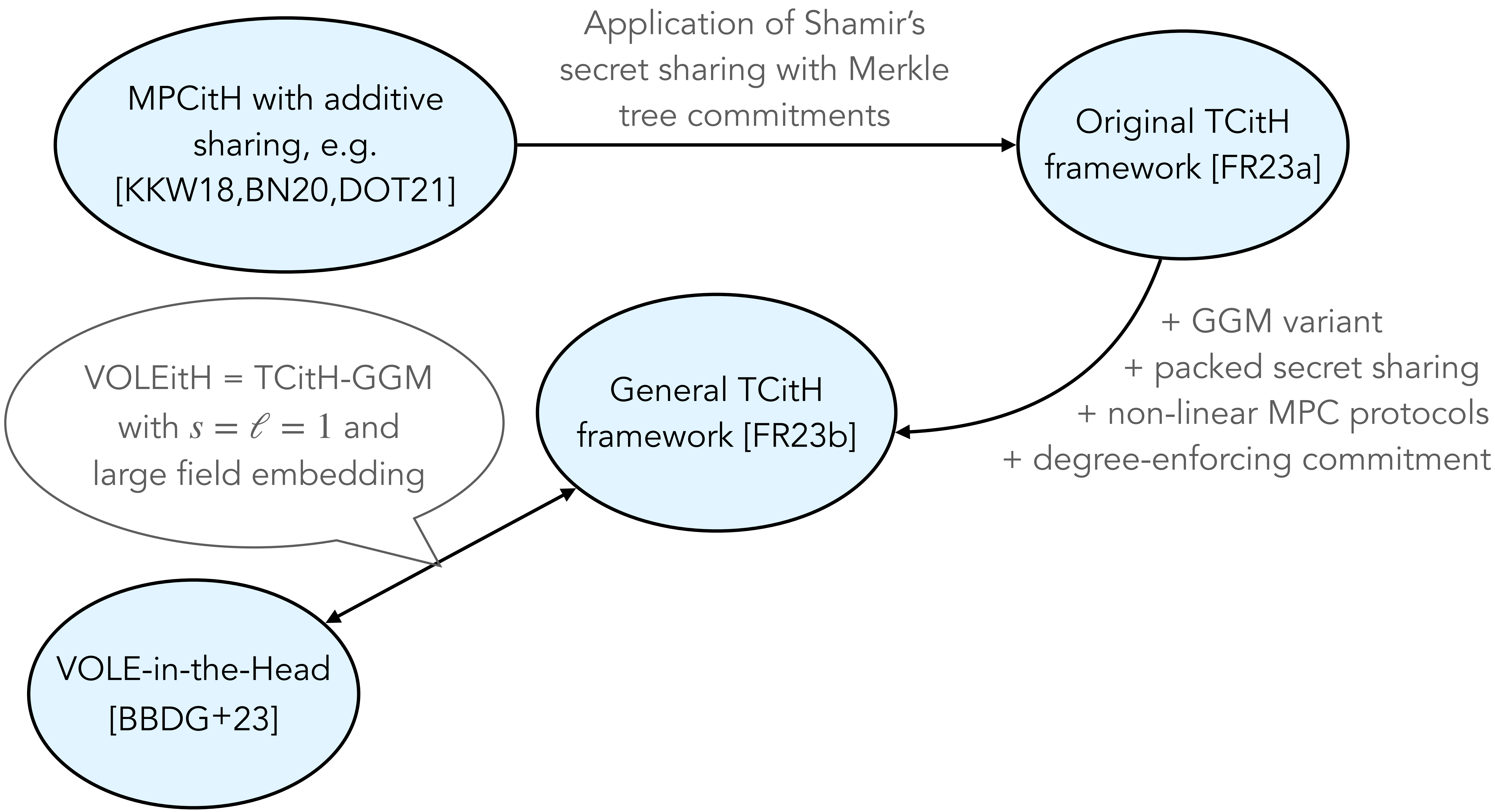

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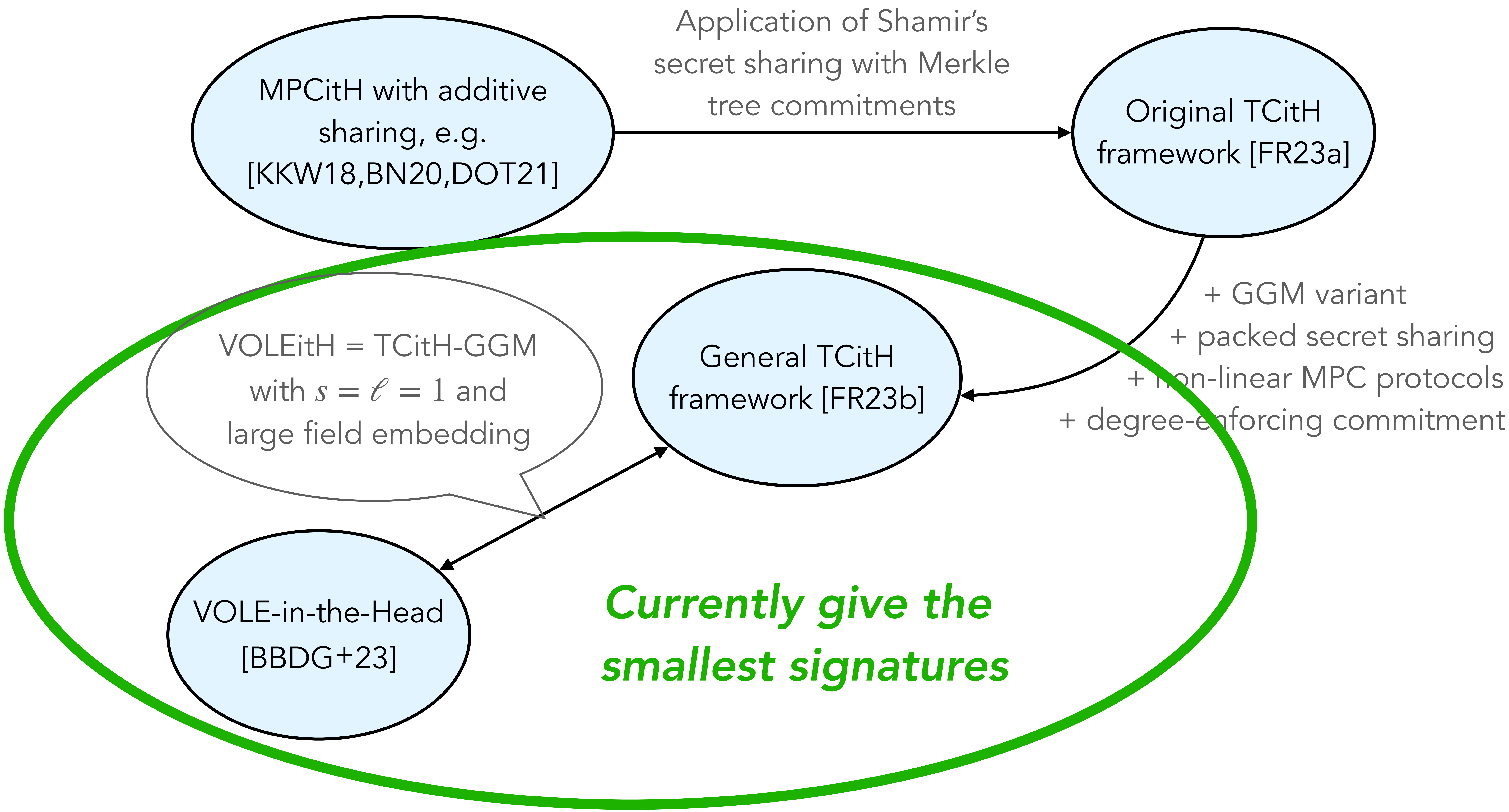

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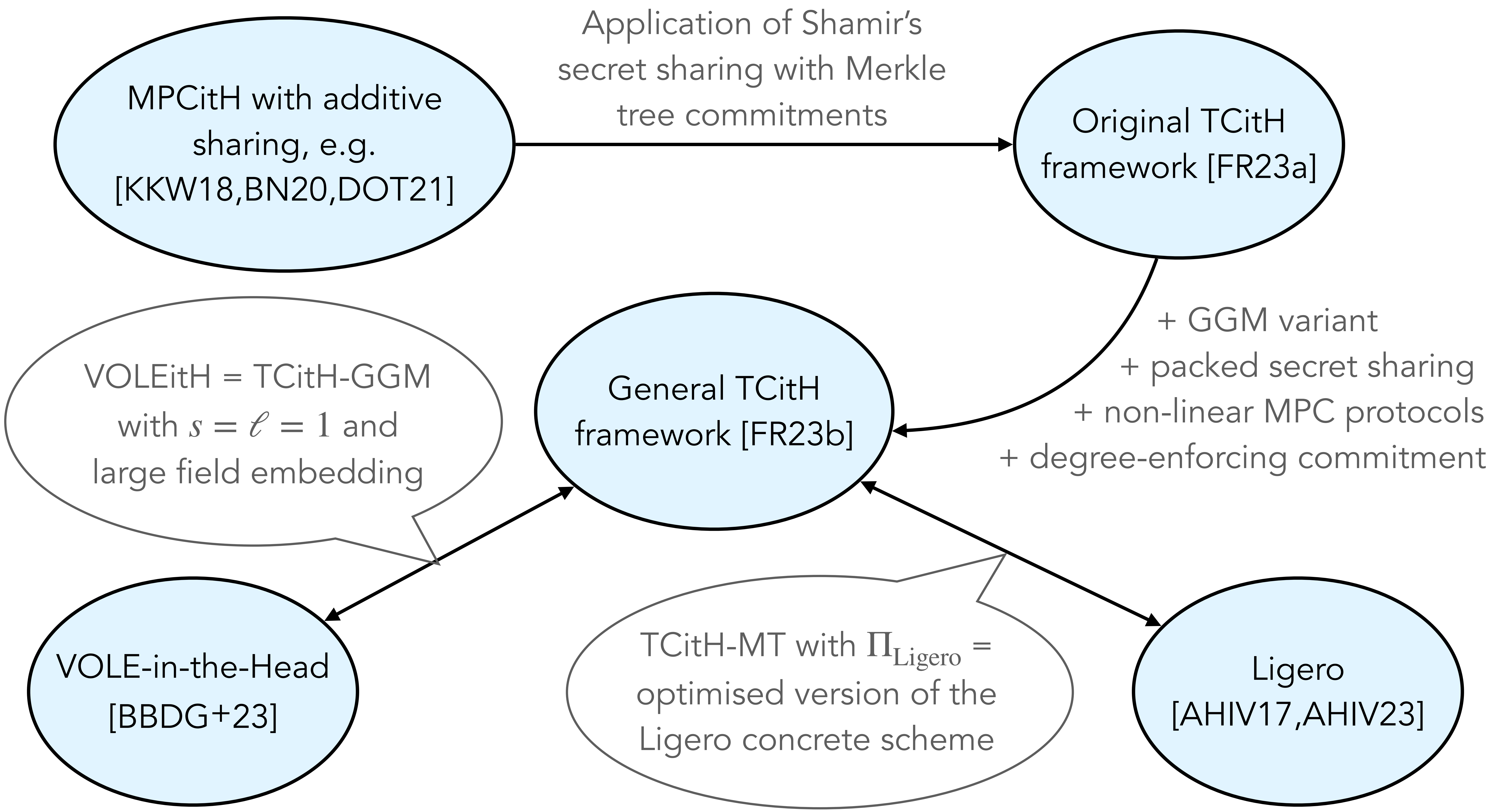

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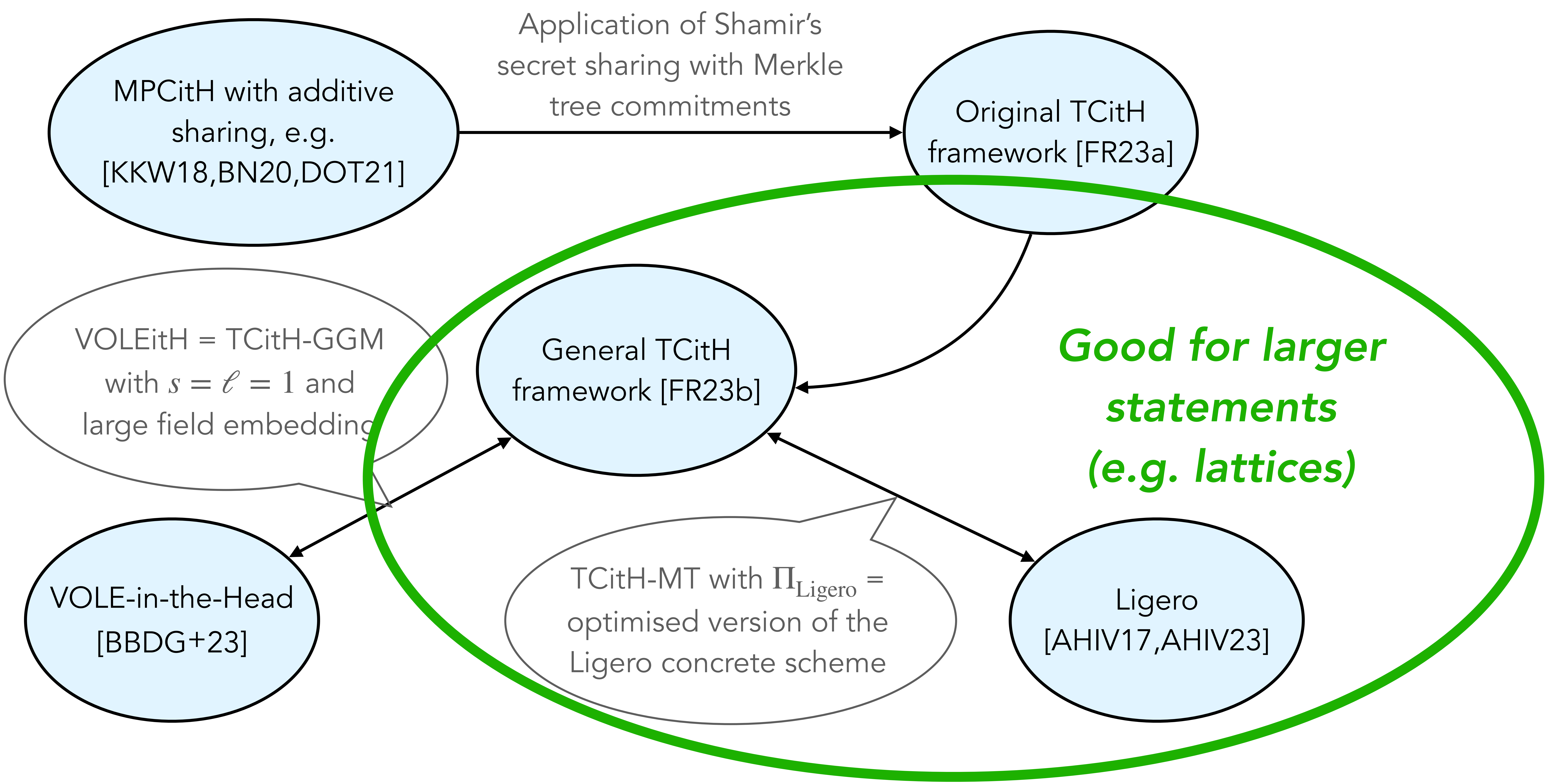

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    - Drastic improvements since 2017
      (in particular thanks to **GGM trees** [KKW18])
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What next?

You find out!